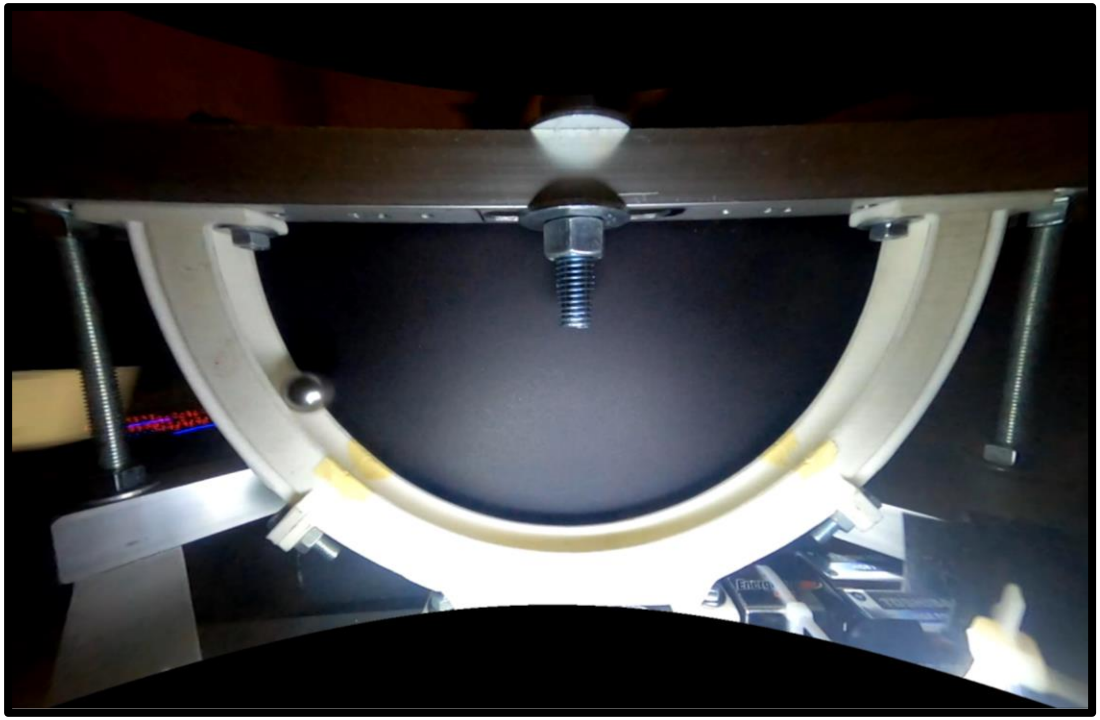
7. Bead Dynamics

A circular hoop rotates about a vertical diameter. A small bead is allowed to roll in a groove on the inside of the hoop. Investigate the relevant parameters affecting the dynamics of the bead.

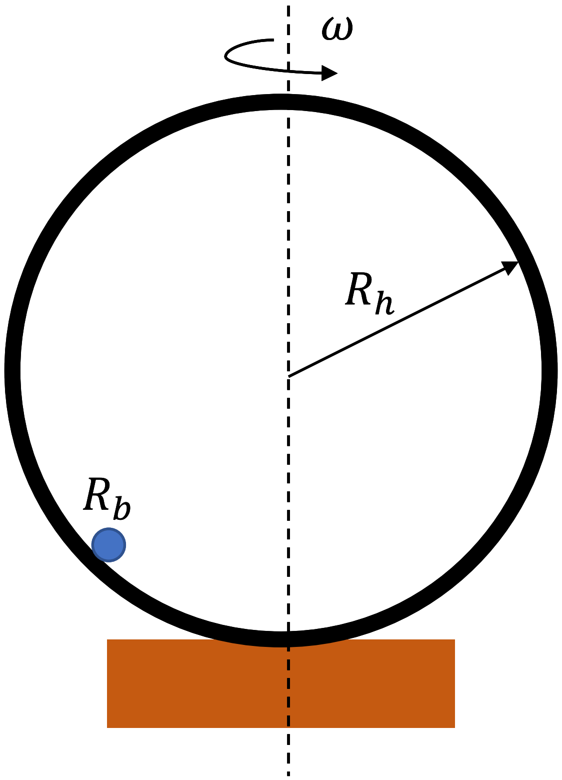


Problem Statement

*A **circular hoop rotates** about a vertical diameter. A small bead is allowed to **roll in a groove** on the inside of the hoop. Investigate the **relevant parameters** affecting the **dynamics of the bead**.*

Parameters:

1. Hoop Radius
2. Bead Radius
3. Hoop Angular Velocity
4. Location of vertical axis
5. Hoop Inclination



Overview

1 **Introduction**
Reproduction of the Phenomenon,
Preliminary Observations

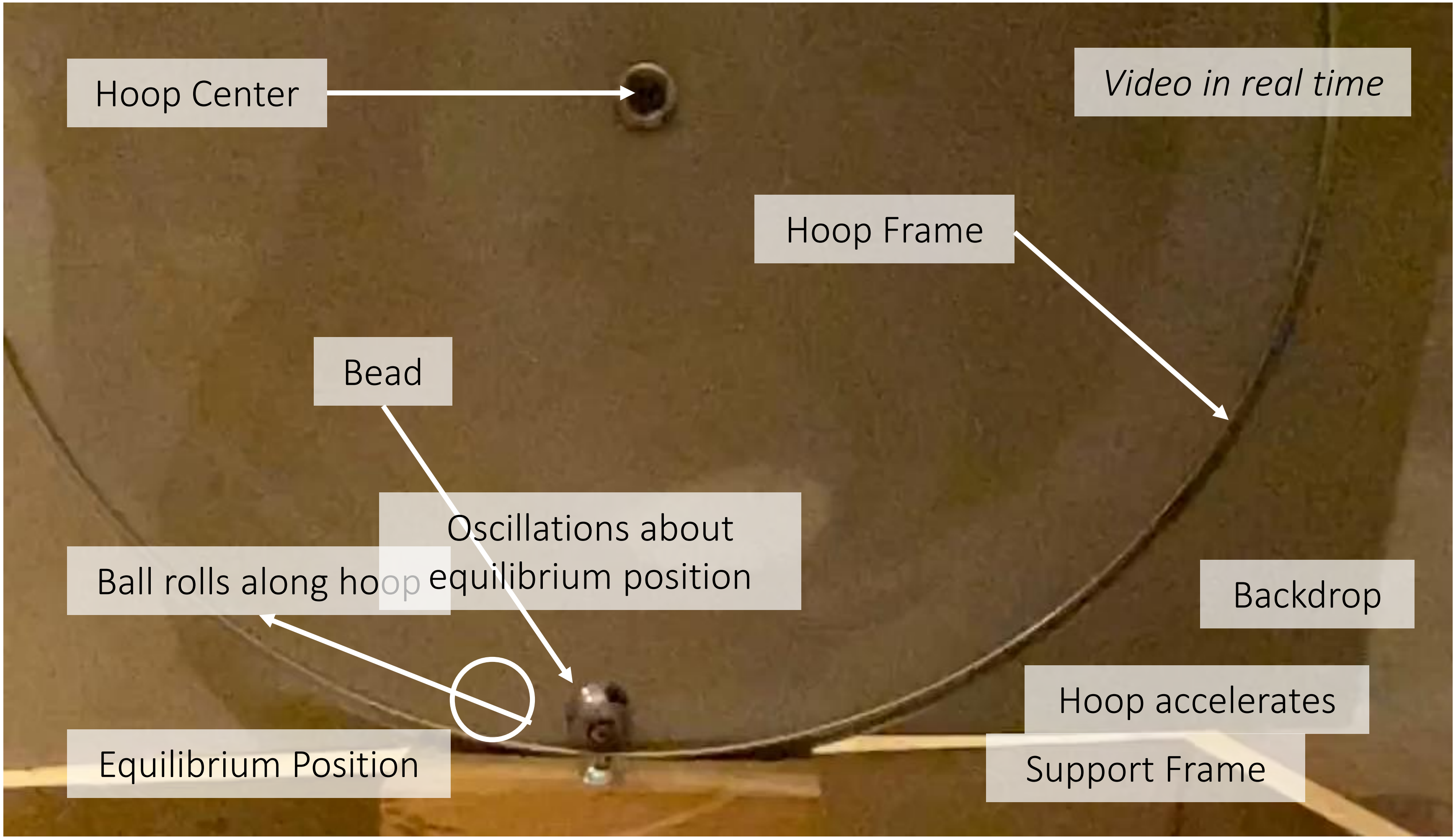
2 **Experimental Setup**
Variable Angle & Offset,
PID Control, Image Analysis

3 **Theoretical Model**
Lagrangian Analysis, Dynamic motion,
Axial Offset, Resonance

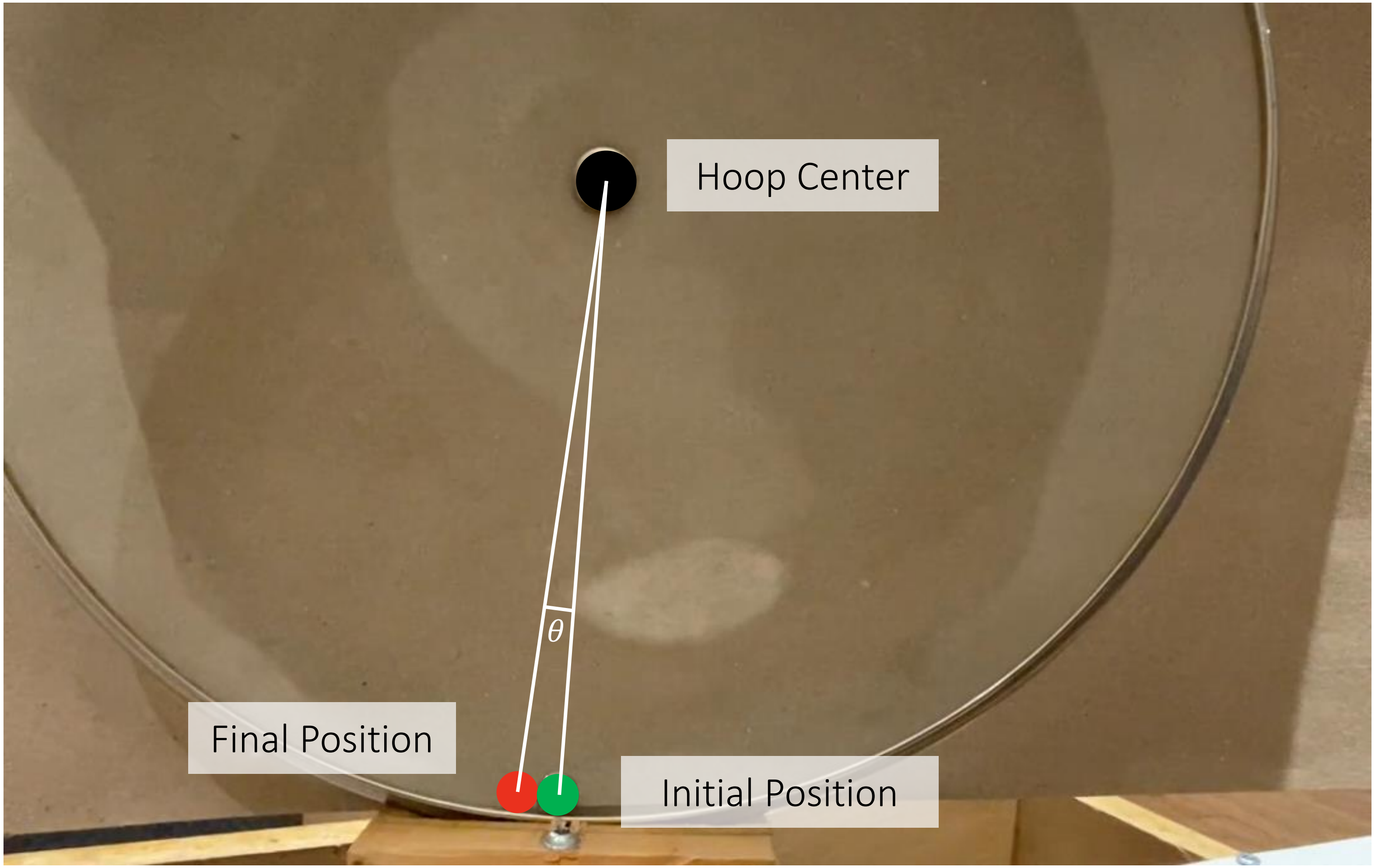
4 **Key Parameter Interactions**
Effects of Changing Physical Parameters

5 **Conclusion**
Further insights, Summary

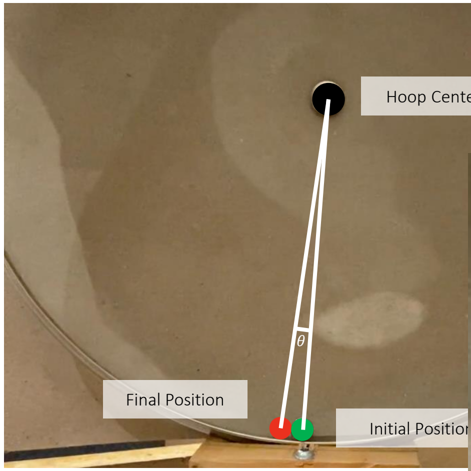
Phenomenon


Hoop Center
Video in real time
Hoop Frame
Bead
Oscillations about equilibrium position
Ball rolls along hoop
Backdrop
Hoop accelerates
Support Frame
Equilibrium Position

Preliminary Observations

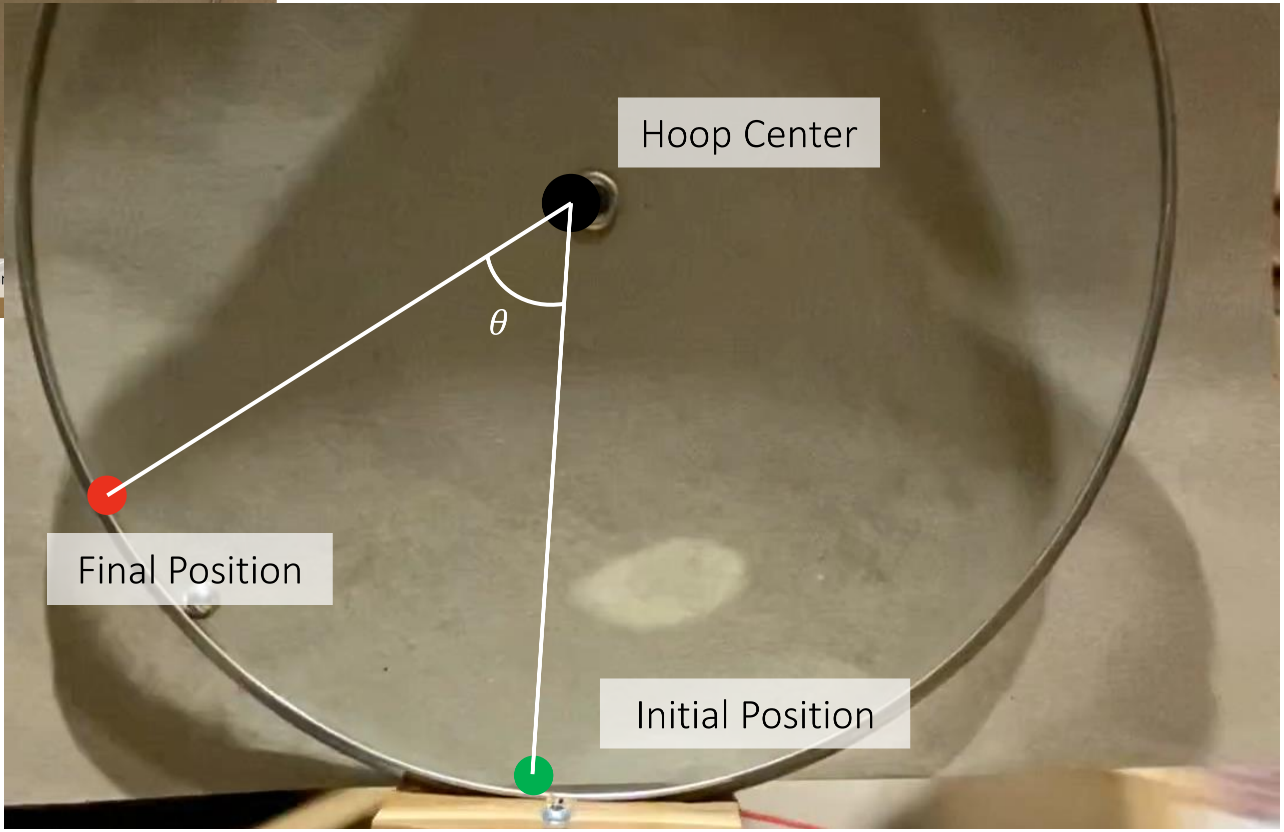


Preliminary Observations

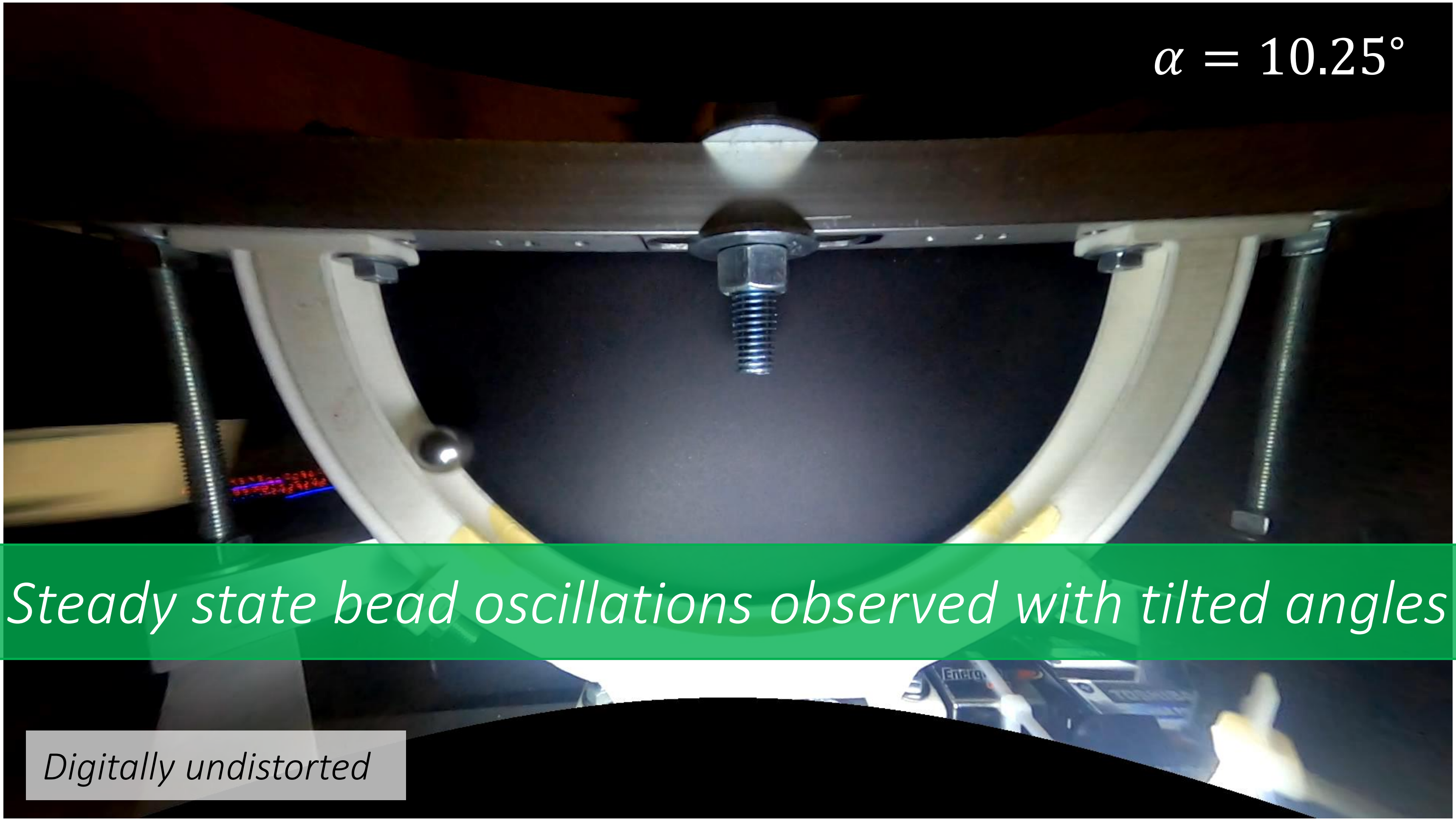


$\omega = 5.2\ \mathrm{rad/s}$

$\omega = 10.5\ \mathrm{rad/s}$



Preliminary Observations with Tilt



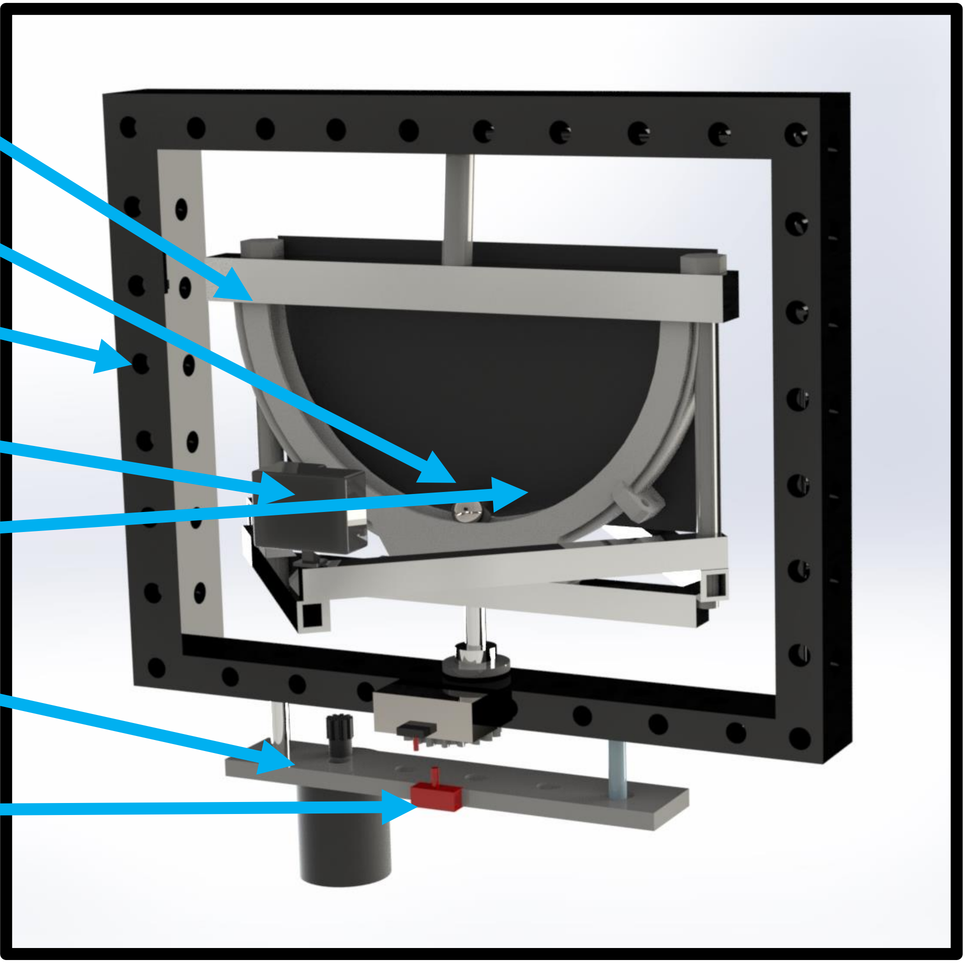
$\alpha = 10.25°$

Steady state bead oscillations observed with tilted angles

Digitally undistorted

Experimental Setup

Experimental Setup



3D printed hoop

Smooth steel bead

Mounted steel frame

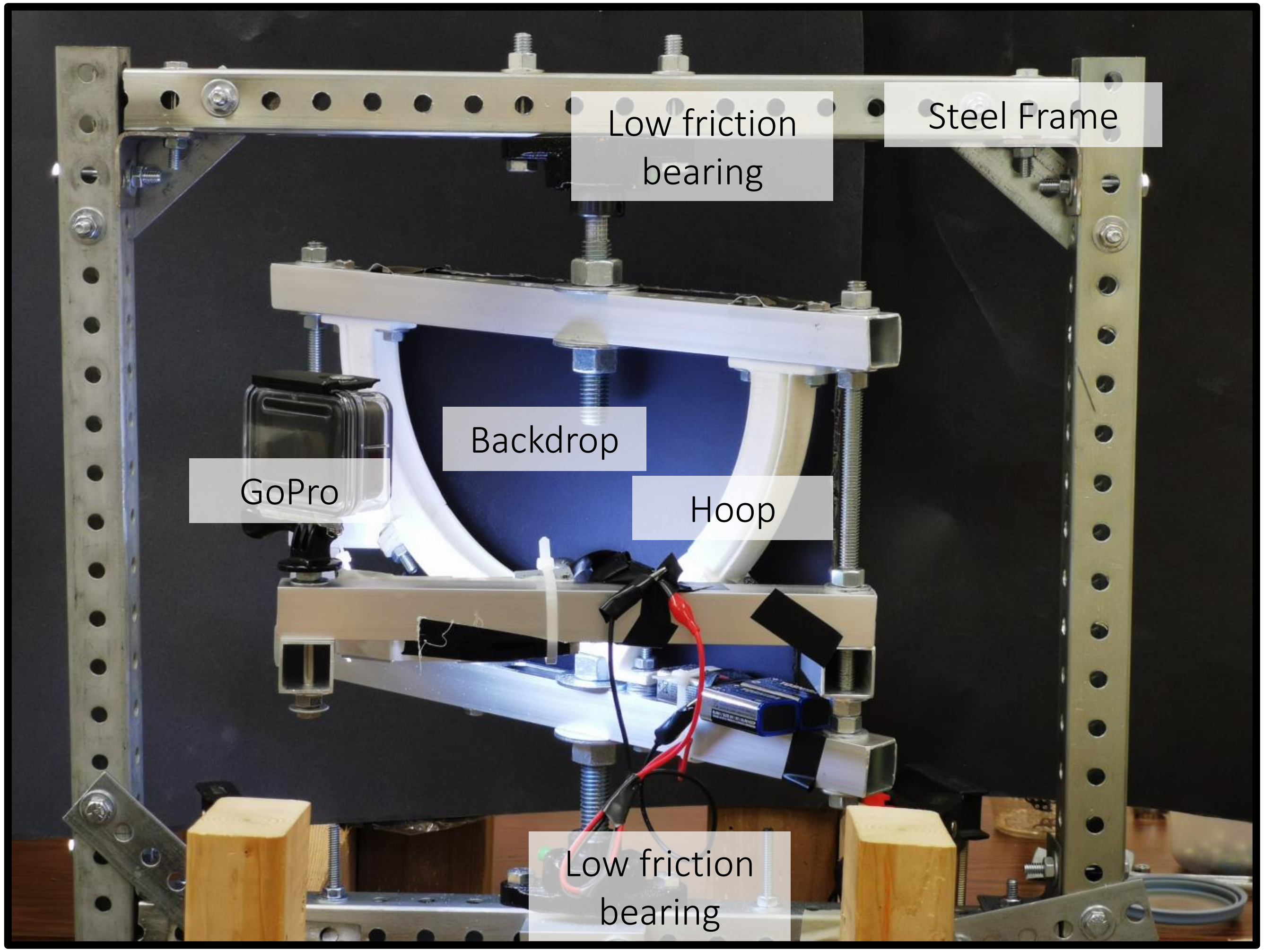
GoProAction Camera

Dark backdrop

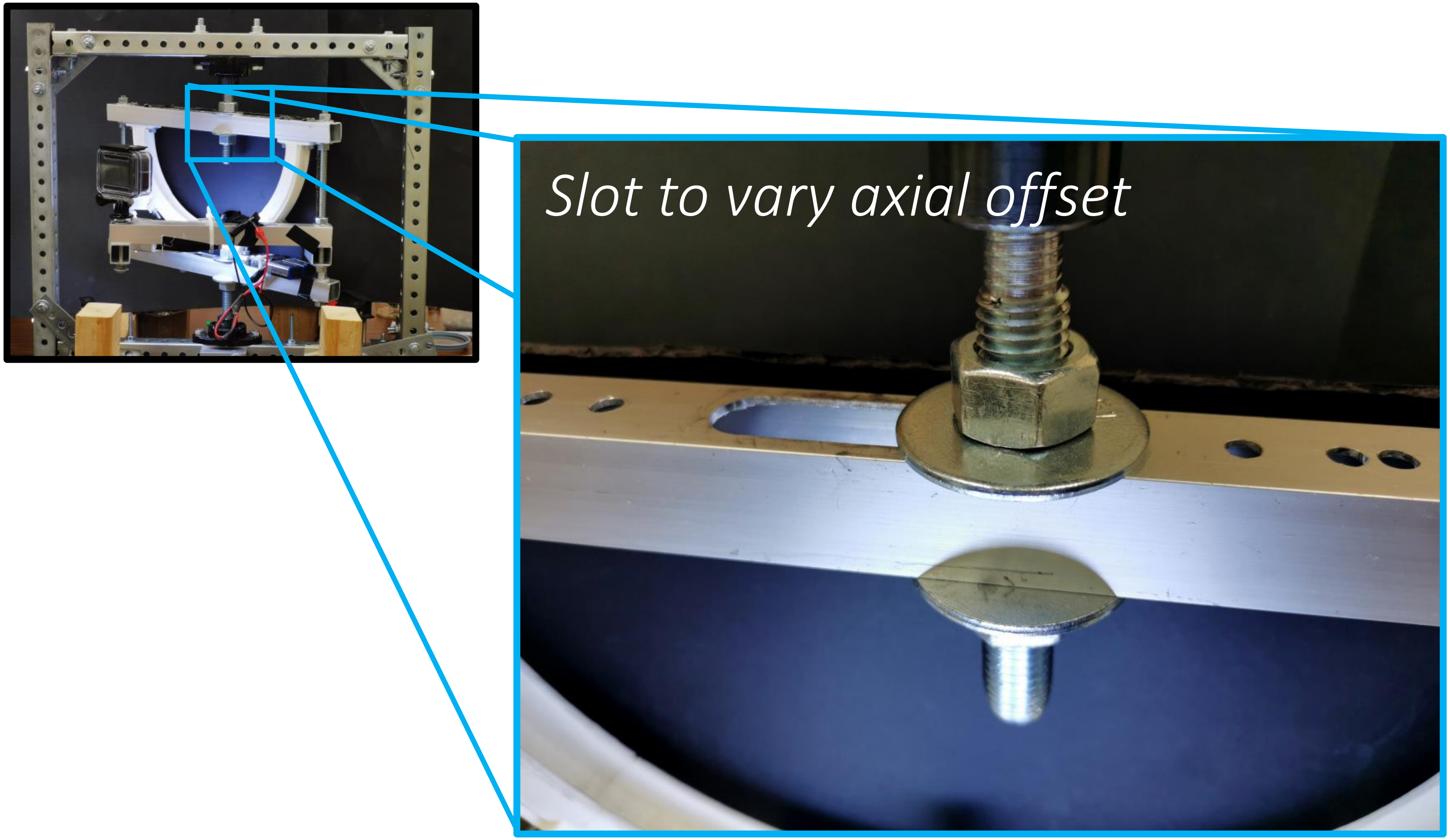
High torque DC Motor

Arduino breakbeam sensors

Experimental Setup

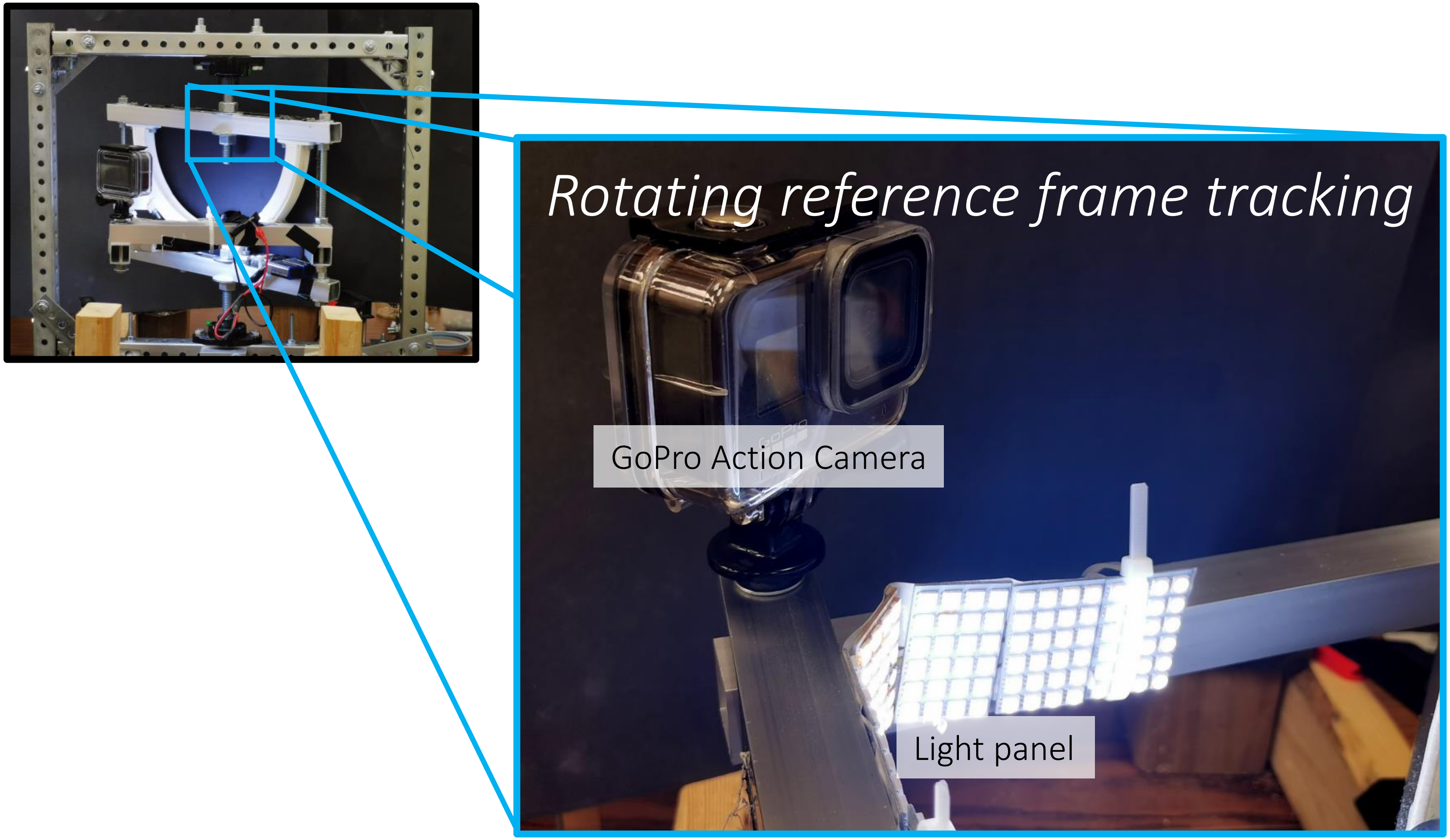


Experimental Setup

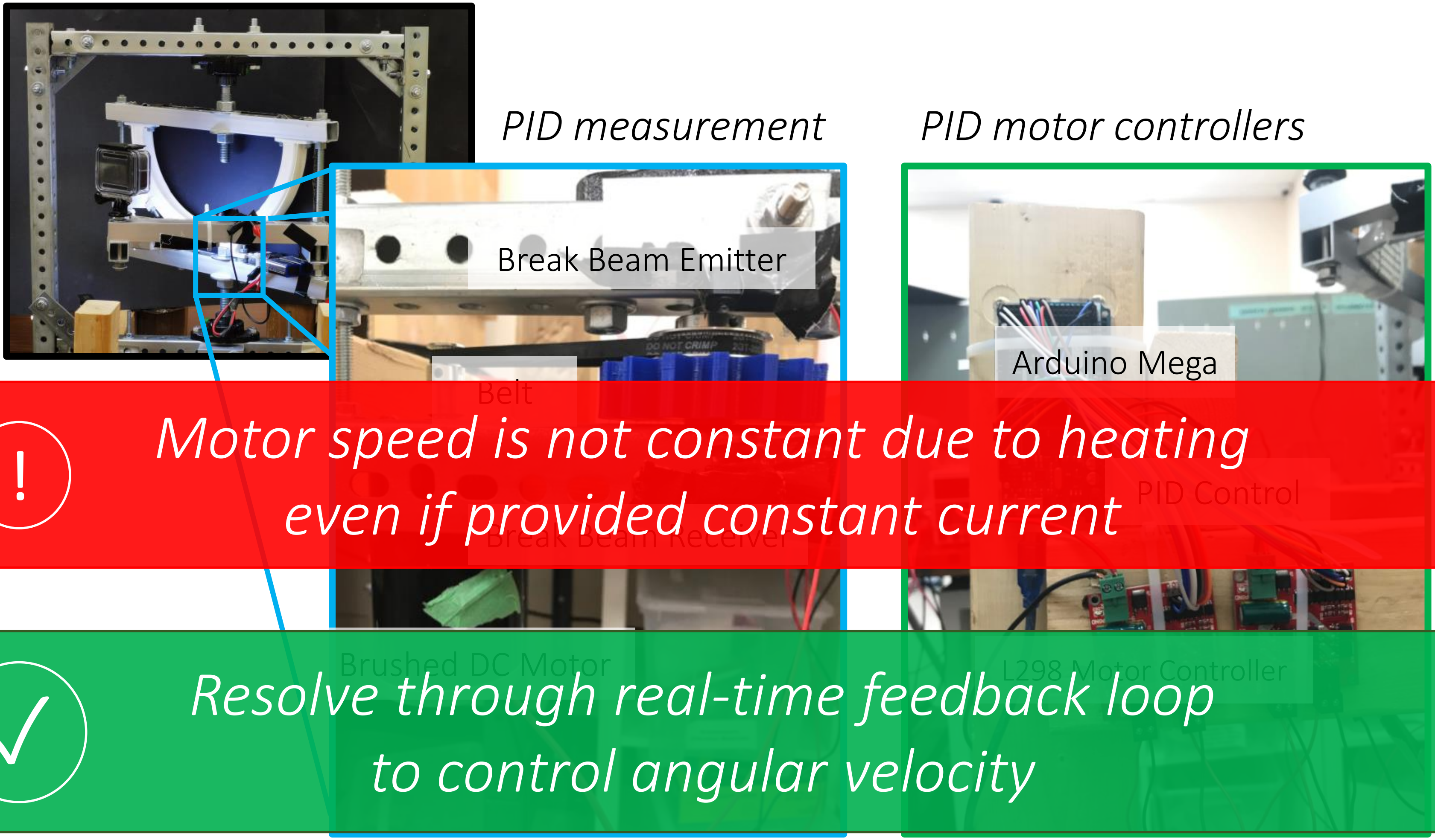


Slot to vary axial offset

Experimental Setup



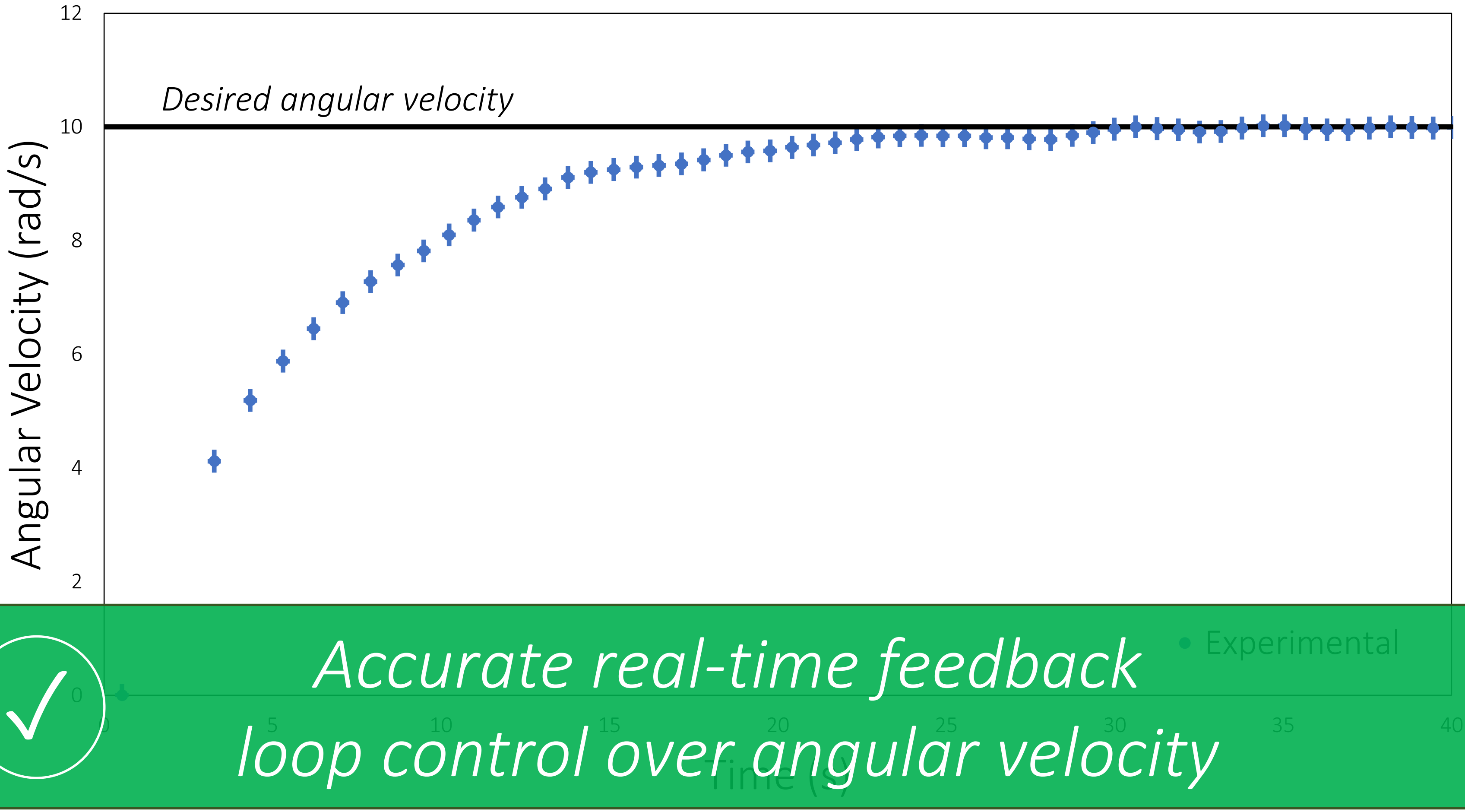
Experimental Setup



! *Motor speed is not constant due to heating even if provided constant current*

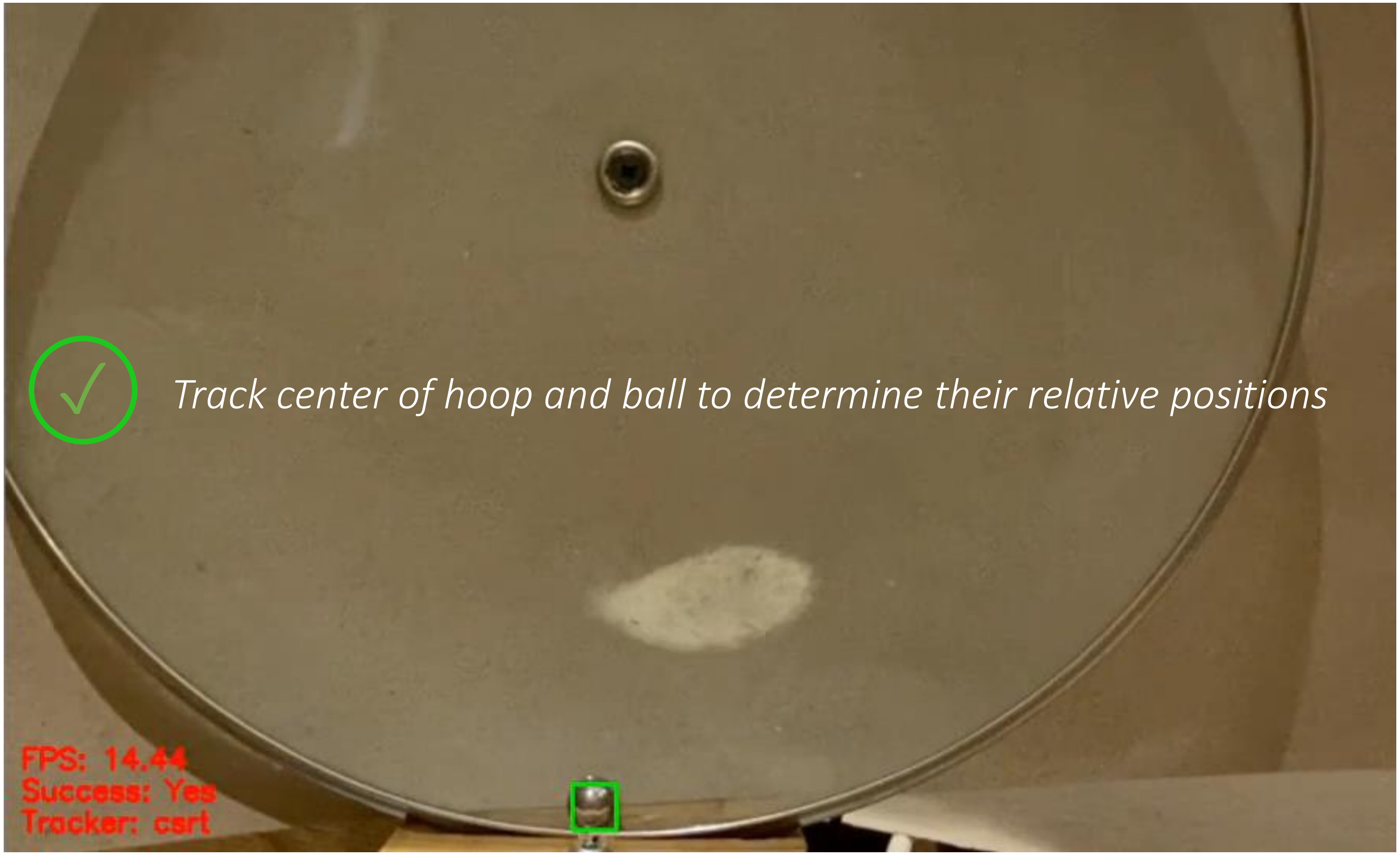
✓ *Resolve through real-time feedback loop to control angular velocity*

PID Control



✓ *Accurate real-time feedback loop control over angular velocity*

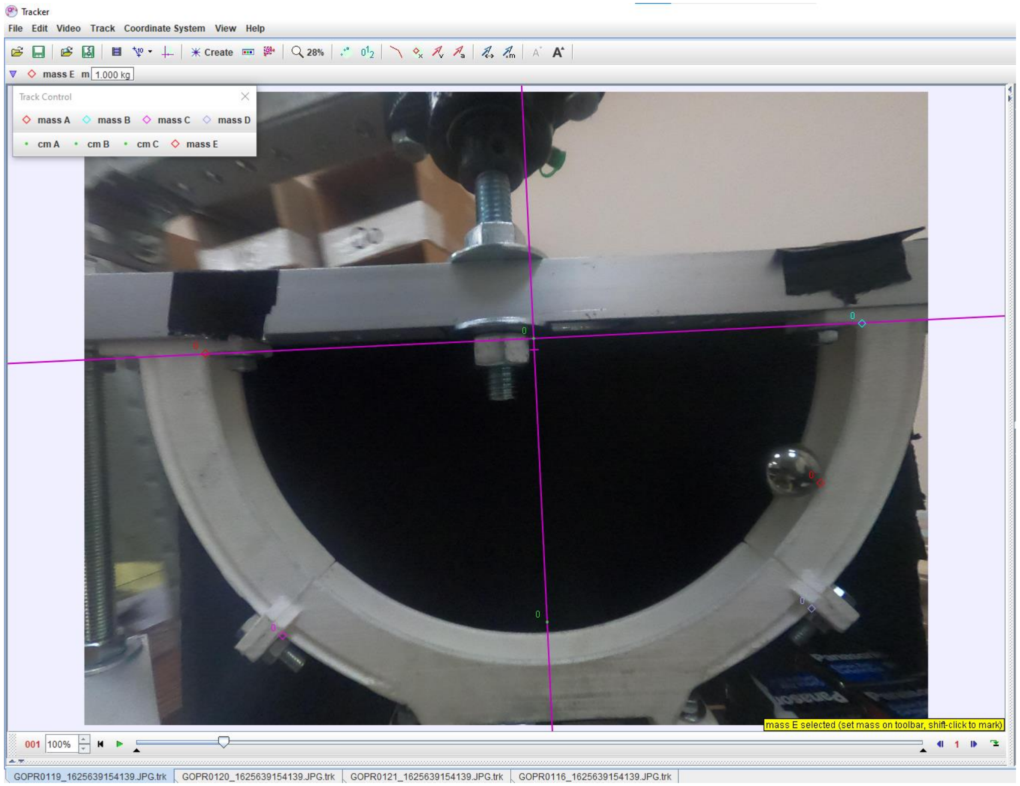
Measurement Method



✓ *Track center of hoop and ball to determine their relative positions*

Measurement Method

Tracker
Video Analysis and Modeling Tool



mass E selected (set mass on toolbar, shift-click to mark)

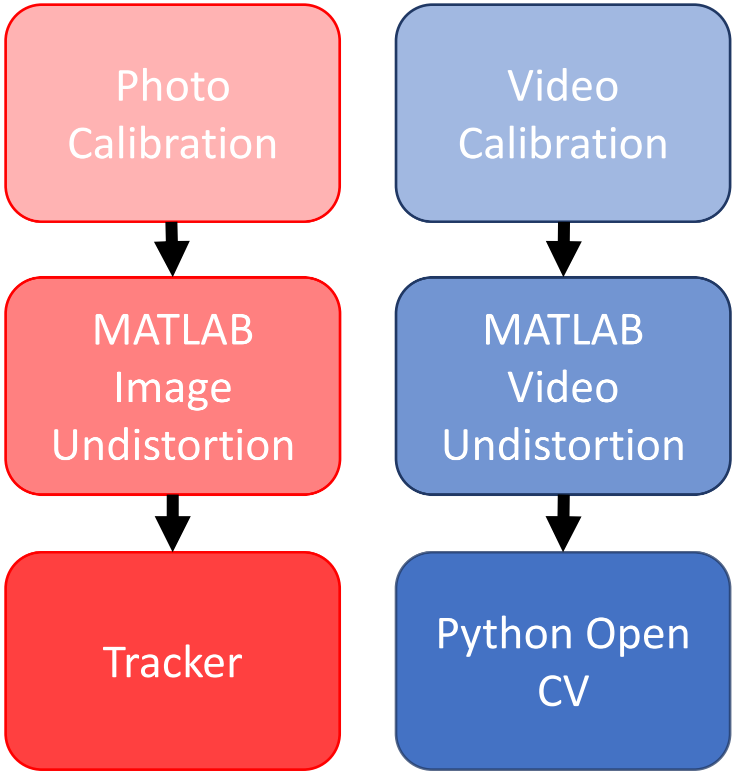


Photo Calibration

MATLAB Image Undistortion

Tracker

Video Calibration

MATLAB Video Undistortion

Python Open CV

Measuring Bead Radius and Mass

Vernier Caliper: $\pm 0.0002 \text{ m}$

Digital Scale: 00.0 – 999.0 ± 0.01 g

$D = 9.5 \pm 0.2 \text{ mm}$

$m_b = 3.56 \pm 0.01 \text{ g}$

$D = 11.1 \pm 0.2 \text{ mm}$

$m_b = 5.59 \pm 0.01 \text{ g}$

$D = 12.7 \pm 0.2 \text{mm}$

$m_b = 8.25 \pm 0.01 \text{ g}$

$D = 14.3 \pm 0.2 \text{mm}$

$m_b = 11.91 \pm 0.01 \text{ g}$

$D = 15.6 \pm 0.2 \text{mm}$

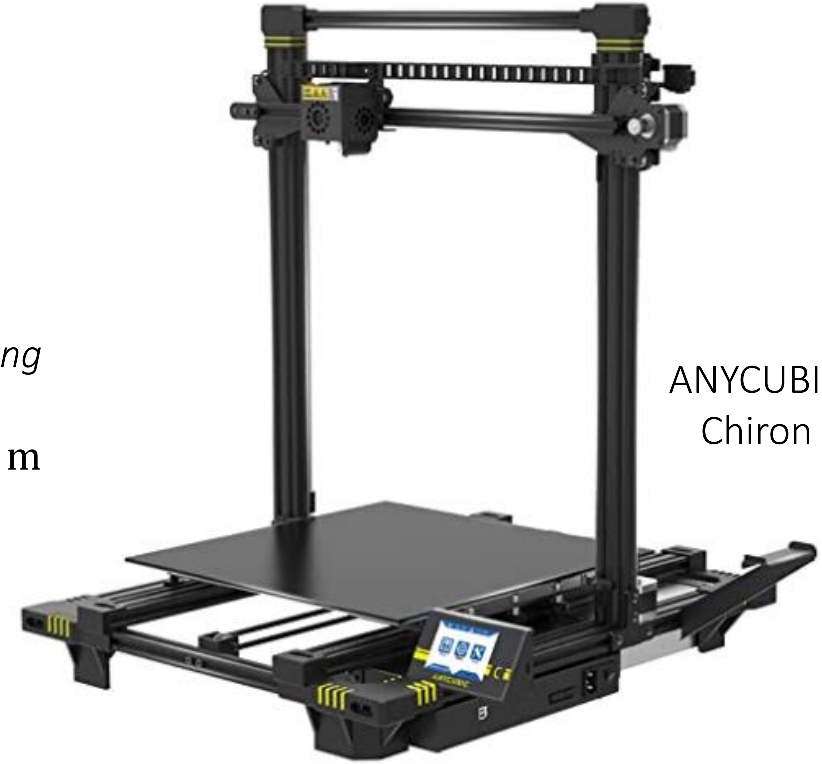
$m_b = 16.36 \pm 0.01 \text{ g}$

✓ *Manufactured Stainless Steel Bearings*

Measuring Physical Parameters

Measuring Tape

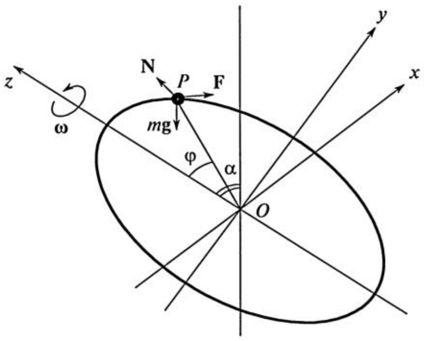
± 0.001 m

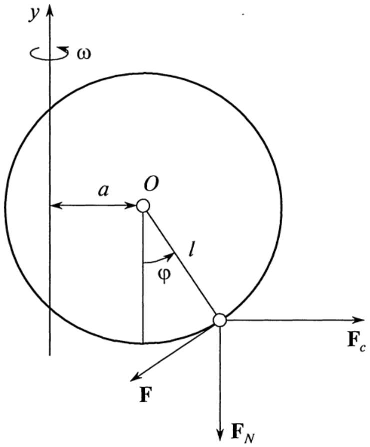


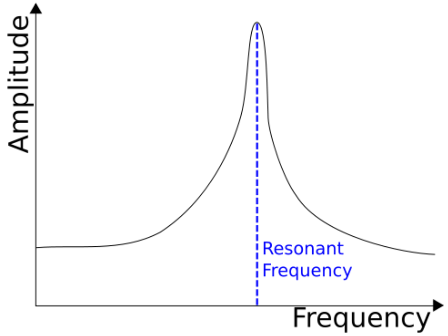
ANYCUBIC Chiron

Theoretical Model

Theoretical Model





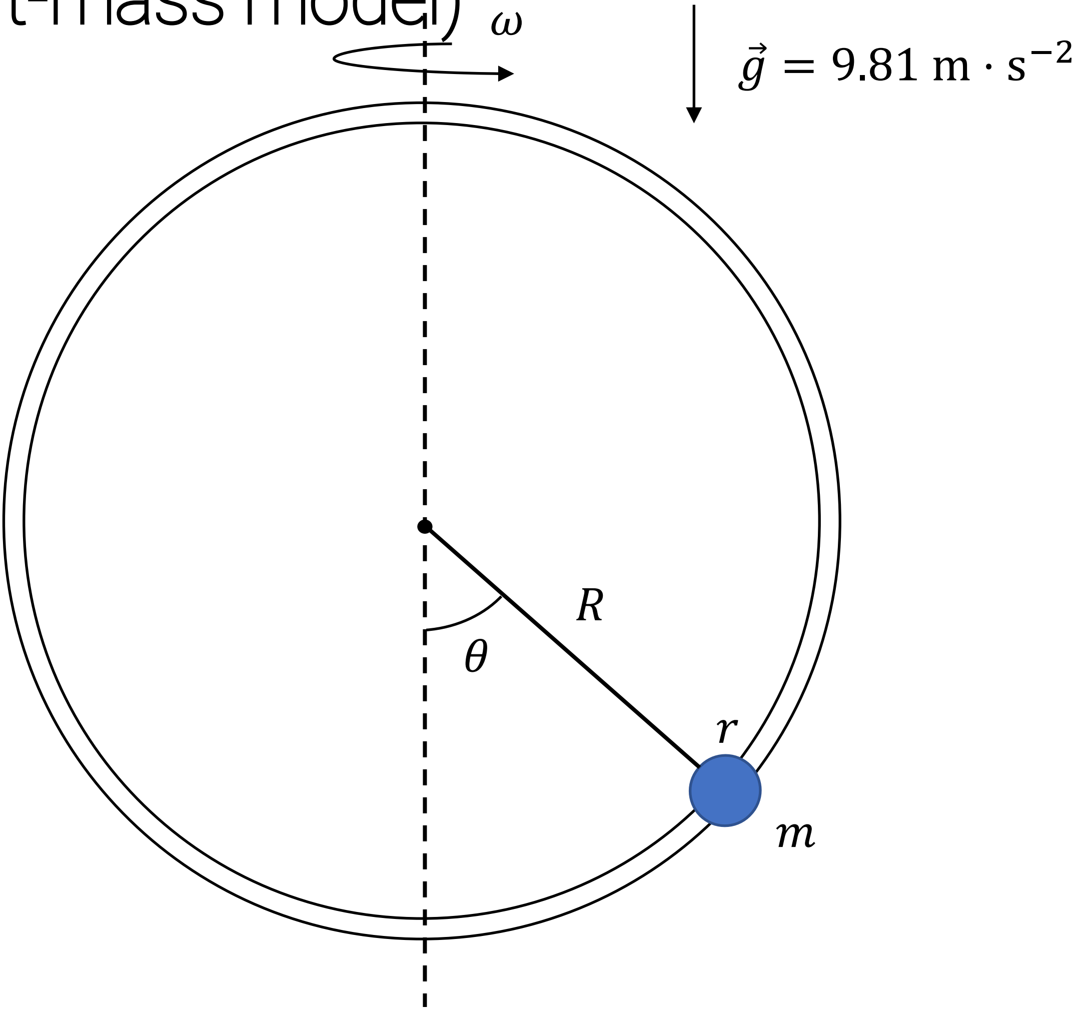


Lagrangian Analysis

Axis of Rotation

Inclination and Resonance

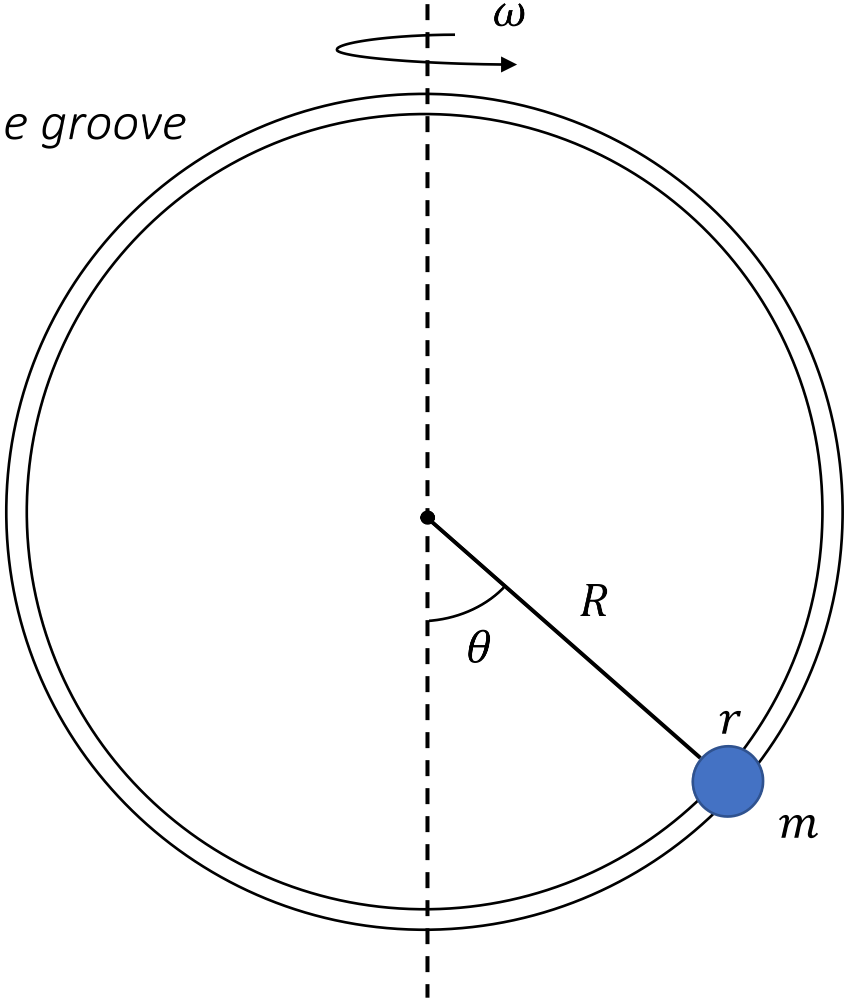
Geometry (point-mass model)



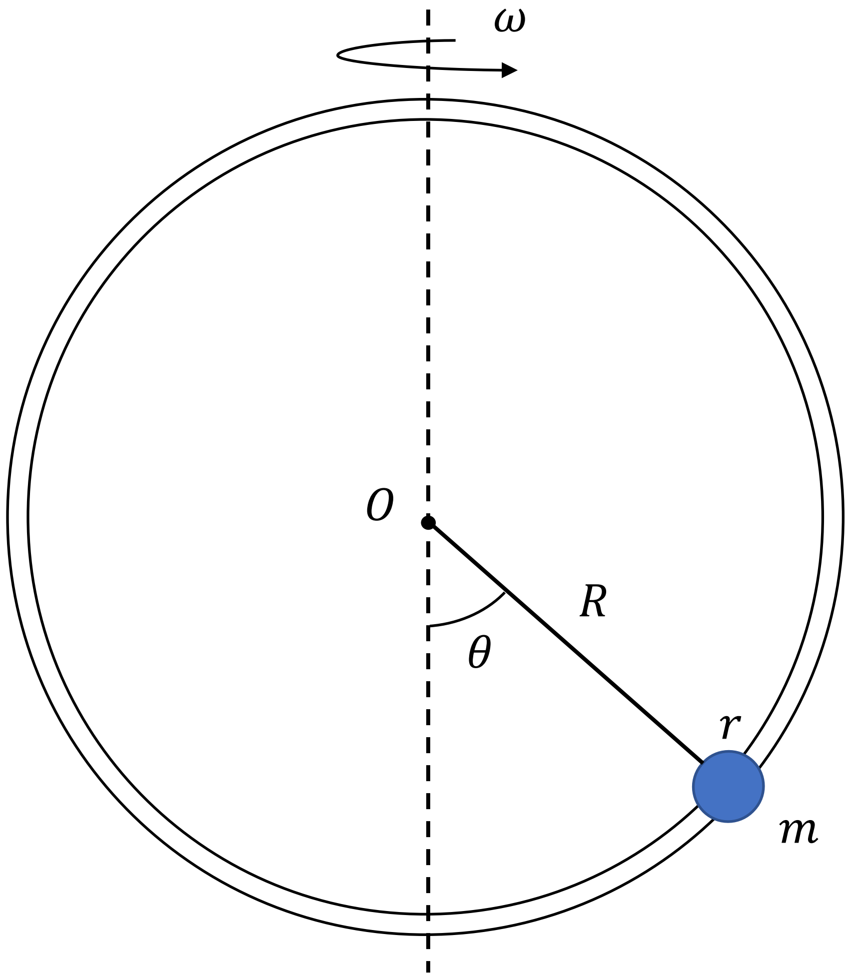
ω: hoop angular velocity
R: hoop radius
r: bead radius
m: bead mass
θ: bead inclination
g: gravitational acceleration

Assumptions

- ✓ *Motion is constrained to along the groove*
  - ✓ *Restrict* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- ✓ *Air resistance is negligible*



Lagrangian Formalism



To simulate the motion, we use Lagrangian Mechanics

$$T = \frac{1}{2} mR^2\left(\dot{\theta}^2 + \sin^2(\theta)\,\omega^2\right)$$

$$U = -mgR(\cos\theta)$$

Euler-Lagrange Equations

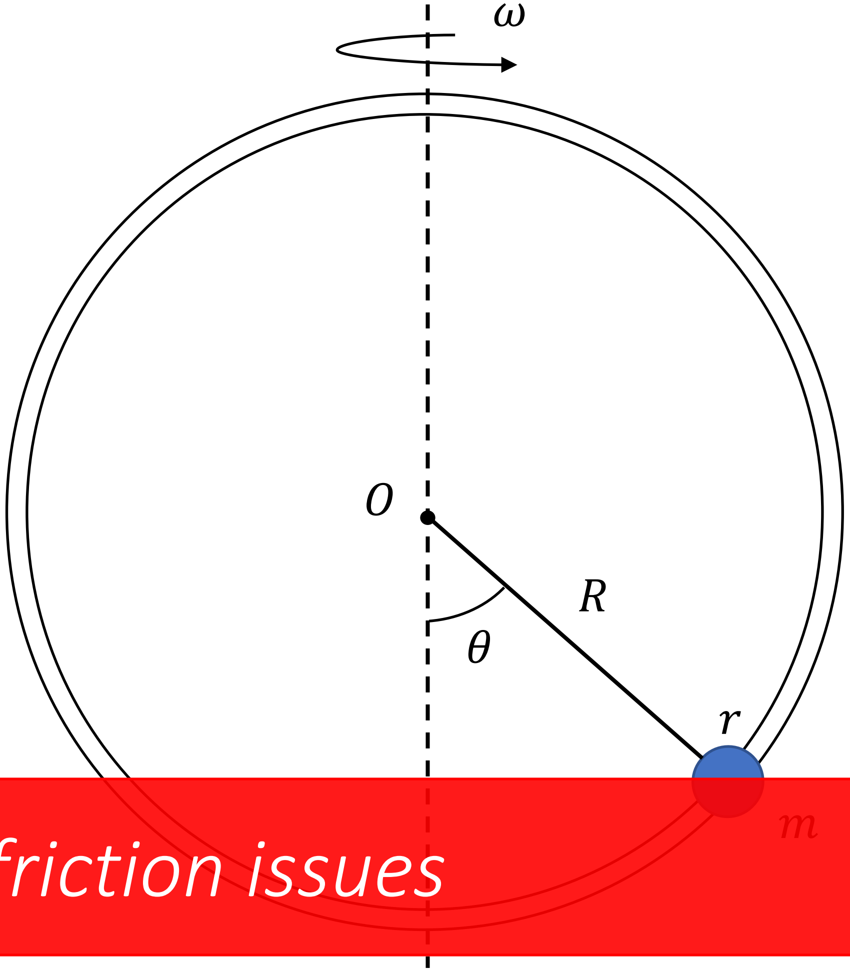
$$\mathcal{L} = \boxed{T} - \boxed{U}$$

Potential Energy

Kinetic Energy

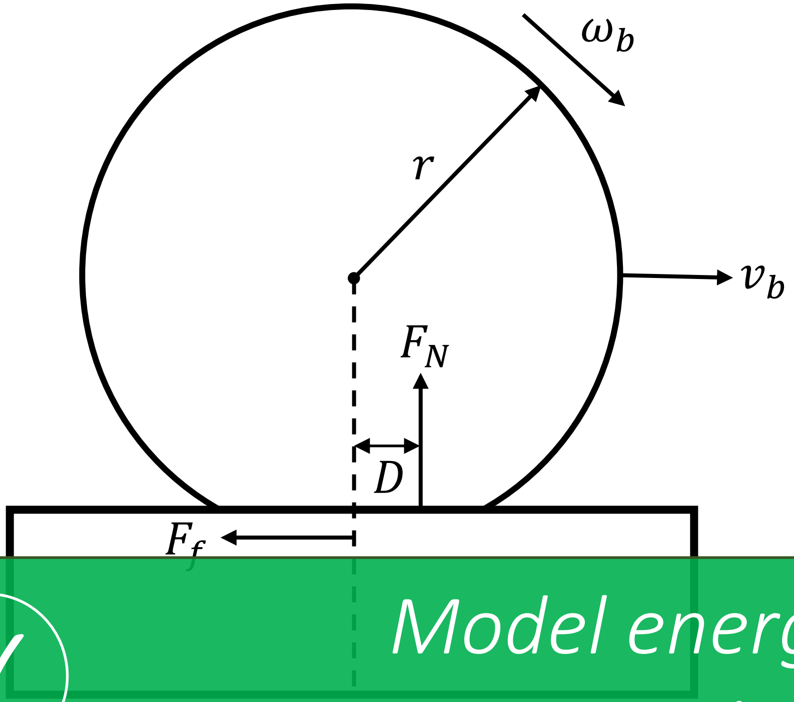
$$T = \frac{1}{2}mR^2\left(\underbrace{\dot{\theta}^2}_{\text{Along Hoop}} + \underbrace{\omega^2\sin^2\theta}_{\text{Rotation}}\right)$$

$$U = -mgR\cos\theta$$



! *Need to resolve friction issues*

Rolling Friction



Equations of Motion

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -mg\sin\theta + F_f$$

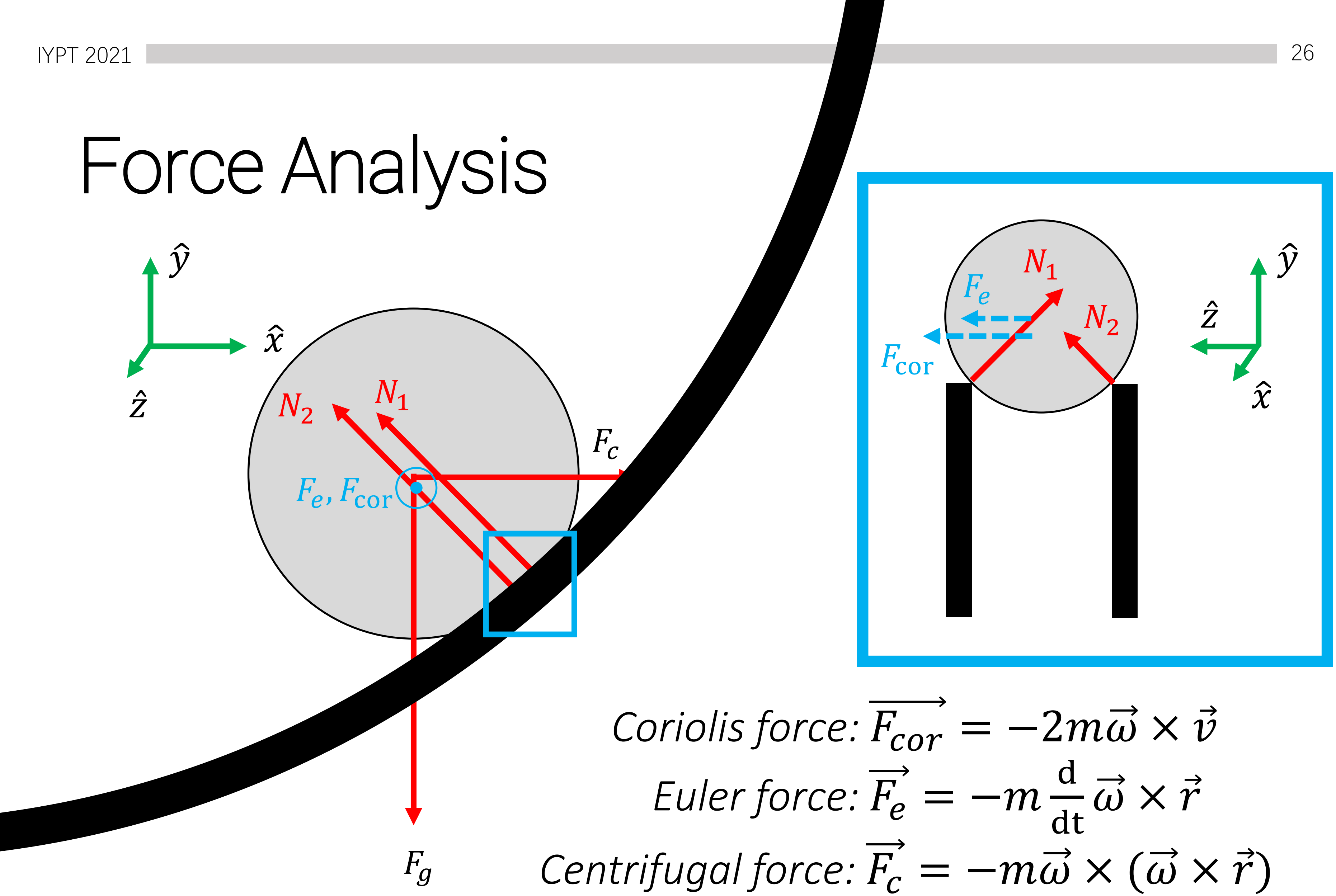
$$F_f r - F_N D = I_{cm}\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

(Cross, 2016)

$$\mu_R = bv$$

✓ *Model energy loss from compression hysteresis*

Force Analysis

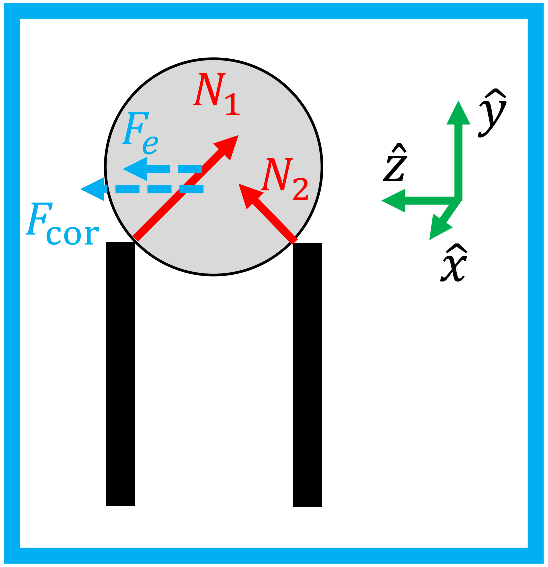


Coriolis force: $\overrightarrow{F_{cor}} = -2m\vec{\omega} \times \vec{v}$

Euler force: $\overrightarrow{F_e} = -m\frac{\mathrm{d}}{\mathrm{dt}}\vec{\omega} \times \vec{r}$

Centrifugal force: $\overrightarrow{F_c} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$

Force Analysis



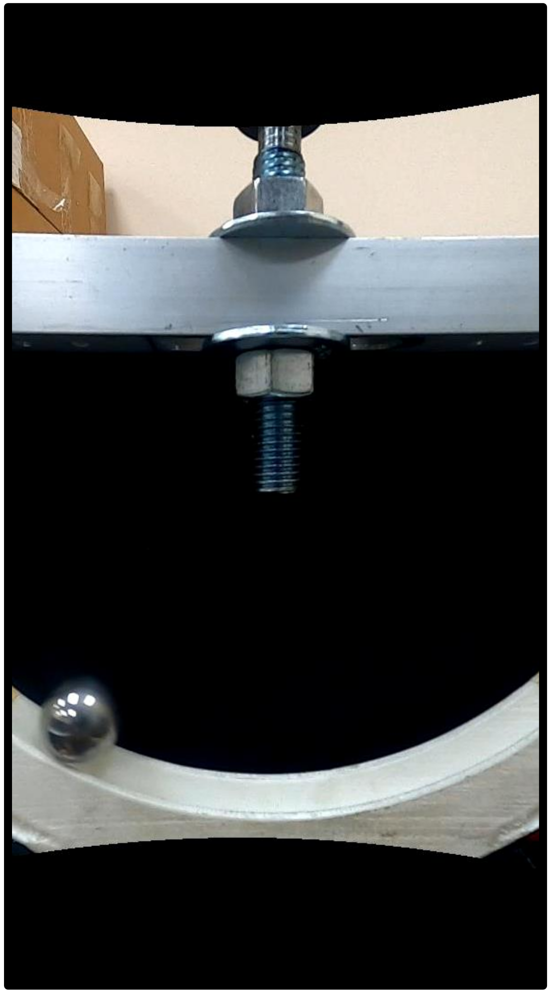
From Newton's 2nd Law:

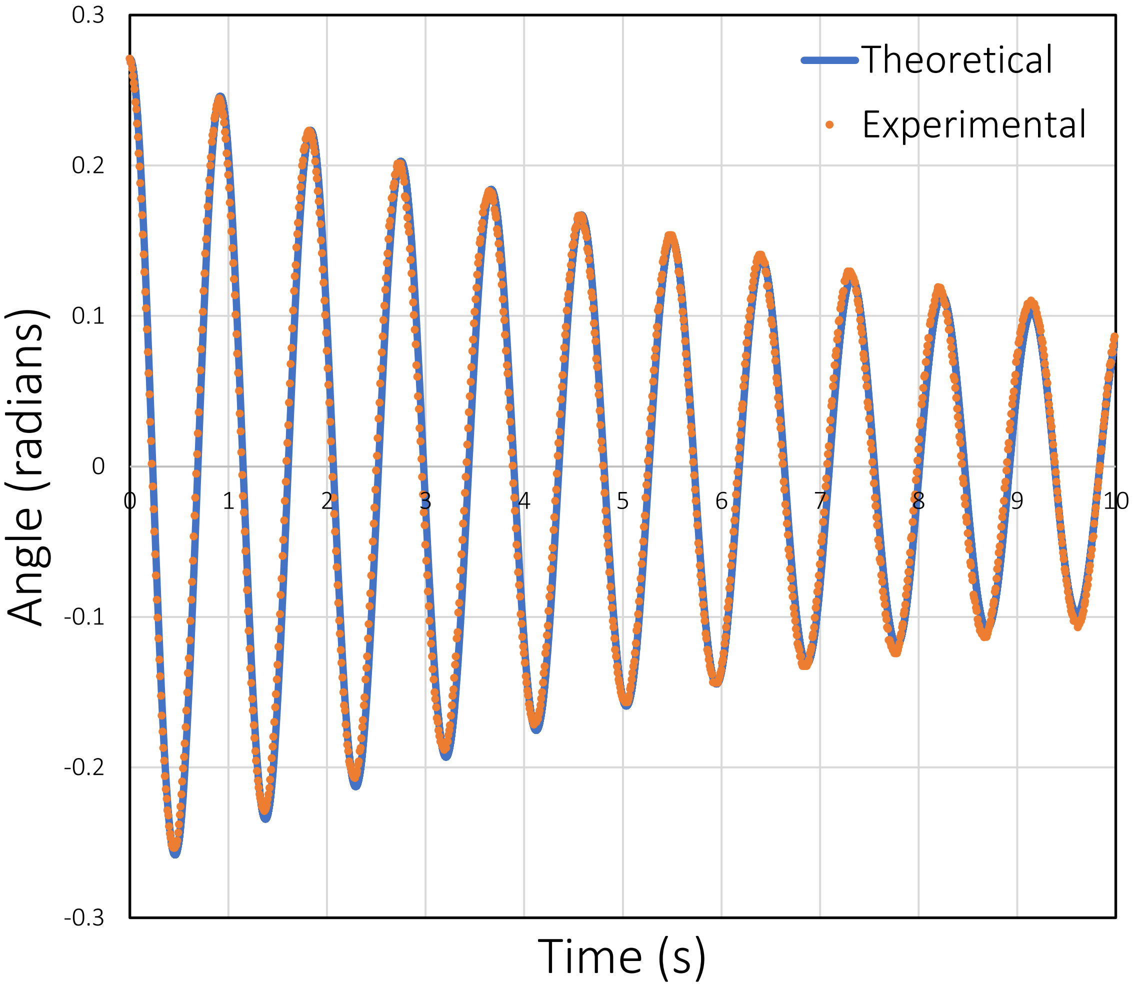
$$N_2 \cos\alpha - N_1 \cos\alpha + F_{cor} + F_e = 0$$
$$N_2 \sin\alpha + N_1 \sin\alpha = mg\cos\theta + F_c \sin\theta$$

$$N_1 = \frac{mg\cos\theta + F_c\sin\theta}{2\sin\alpha} + \frac{F_{cor} + F_e}{2\cos\alpha}$$
$$N_2 = \frac{mg\cos\theta + F_c\sin\theta}{2\sin\alpha} - \frac{F_{cor} + F_e}{2\cos\alpha}$$

Coriolis and Euler force create a difference between normal forces

Experimental Friction Fit





Euler-Lagrange Equations

$$T = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2 \theta \right)$$

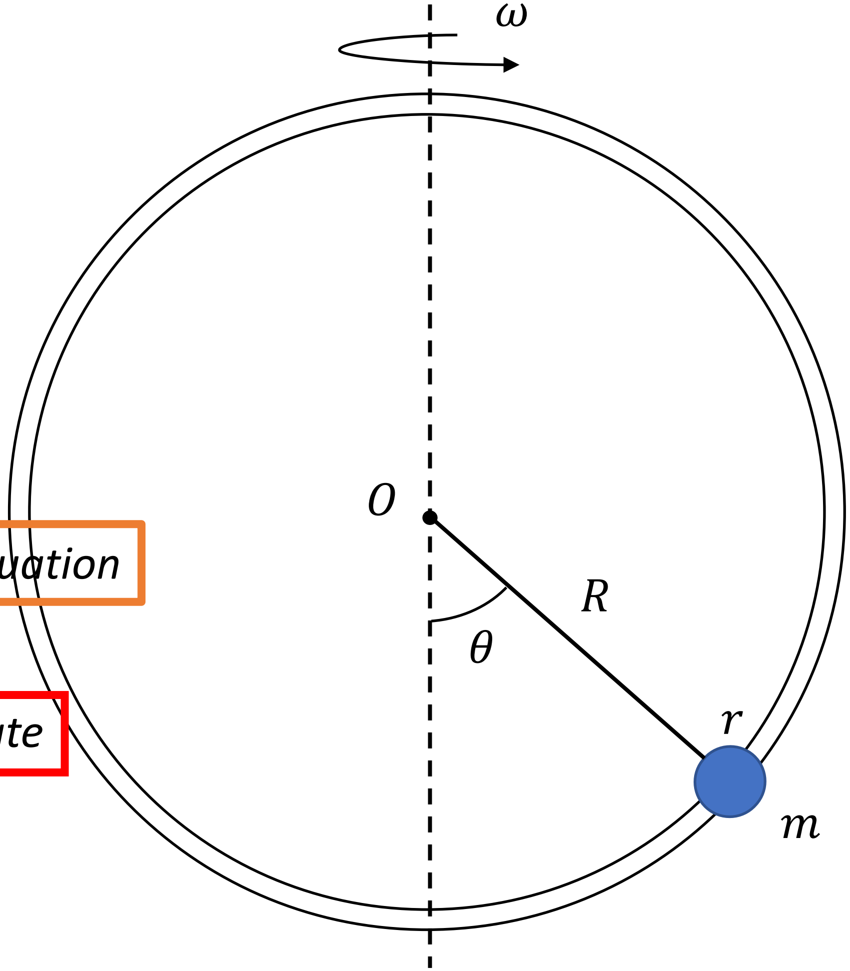
$$U = -mgR\cos\theta$$

$$Q_\theta = -bR^2\dot{\theta}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \boxed{\theta}}}$$

Euler-Lagrange Equation

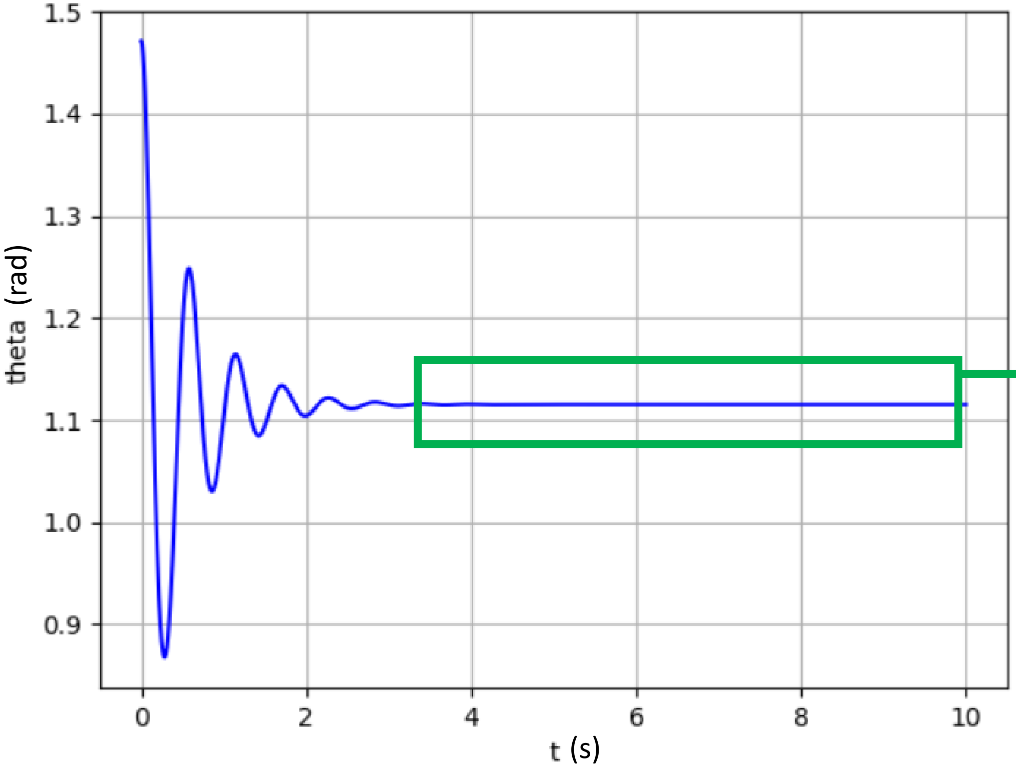
Generalized Coordinate



Combined Solution

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} \longrightarrow \ddot{\theta} + \frac{b}{m}\dot{\theta} + \left(\frac{g}{R} - \omega^2 \cos\theta\right)\sin\theta = 0$$

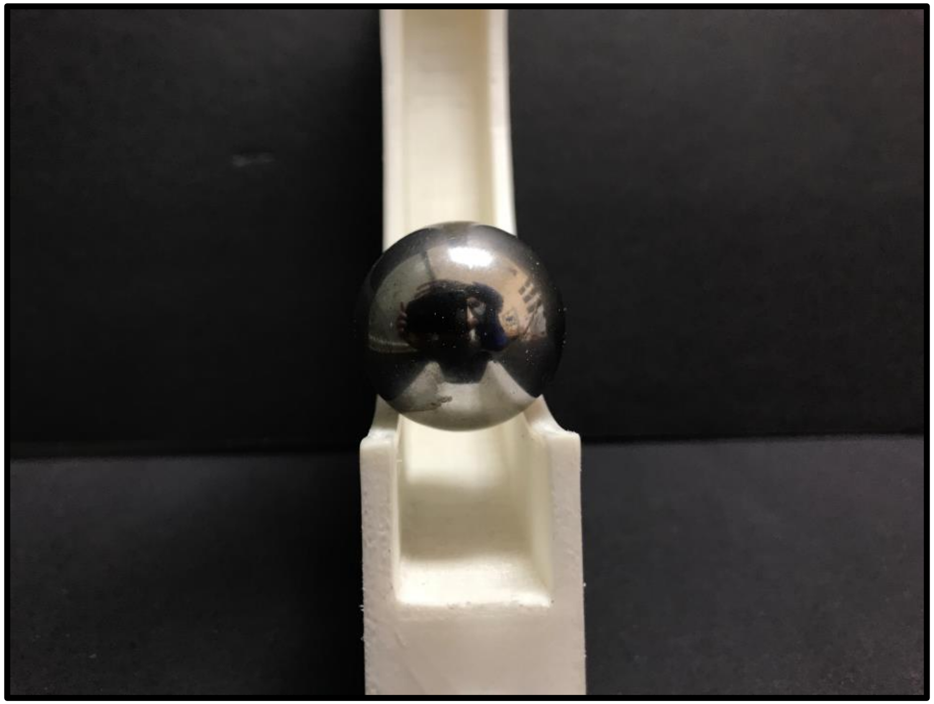
$$\omega = 4\pi, b = 0.1, m = 0.028, R = 0.141$$

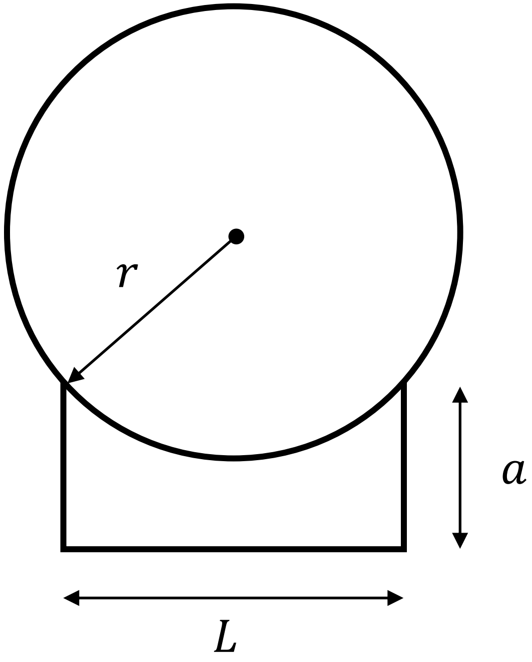


Simulated with scipy odeint

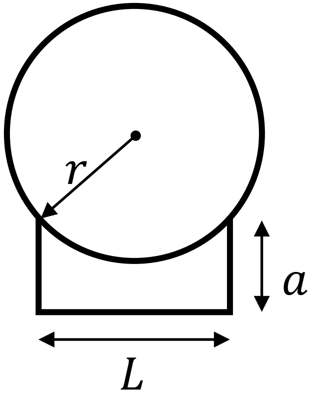
Equilibrium Solution Exists

Groove and Sphere Correction





Groove and Sphere Correction



$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)-\frac{\partial\mathcal{L}}{\partial\theta}\longrightarrow\cancel{\ddot{\theta}+\frac{b}{m}\dot{\theta}+\left(\frac{g}{R}-\omega^2\cos\theta\right)\sin\theta=0}$$

$$\ddot{\theta}+\kappa\left[\frac{b}{m}\dot{\theta}+\left(\frac{g}{R_{CM}}-\omega^2\cos\theta\right)\sin\theta\right]=0$$

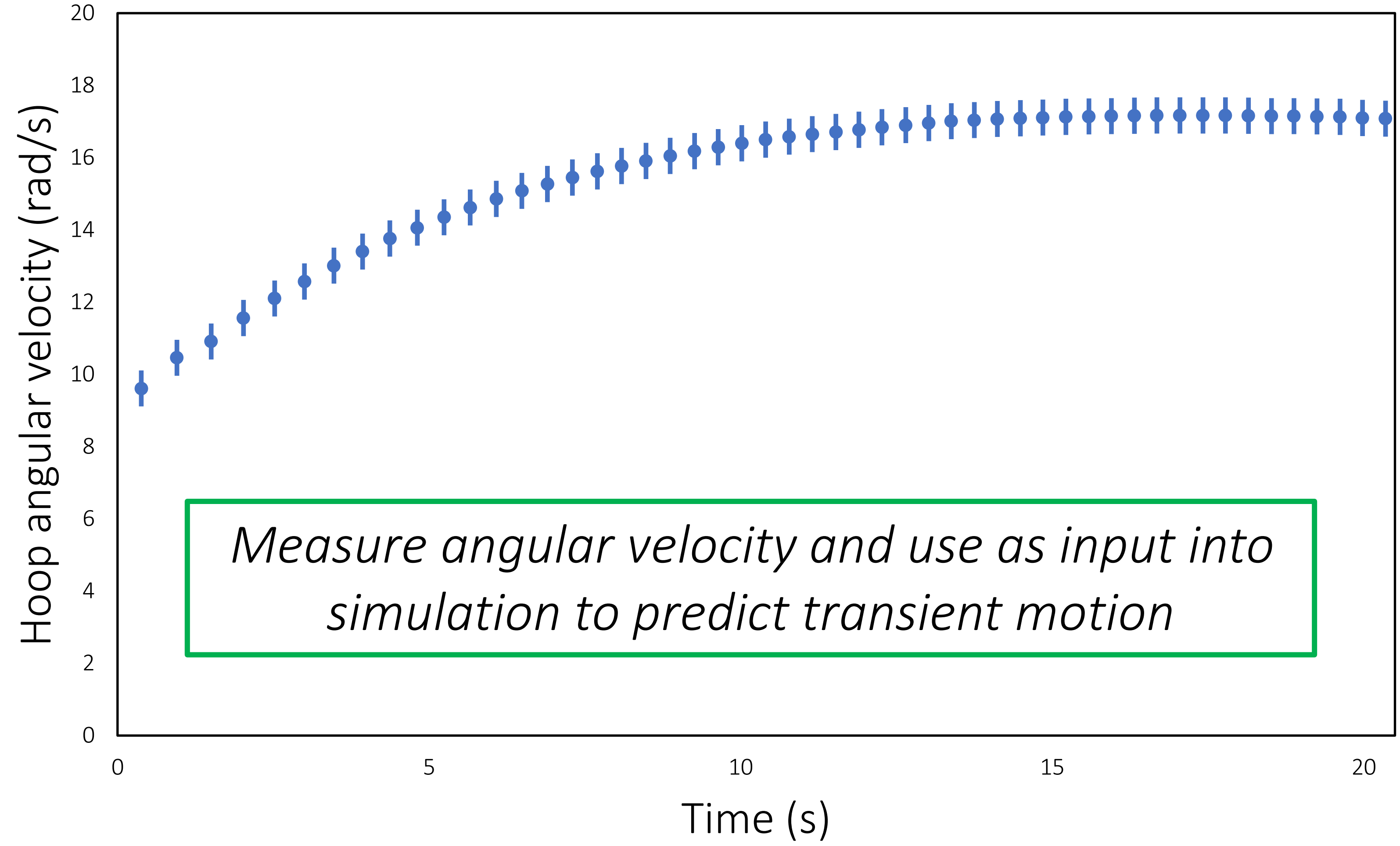
where

$$\kappa=\left[1+\gamma\frac{r^2}{R_{CM}^2}\left(\frac{R_{CM}}{\sqrt{r^2-\frac{L^2}{4}}}+1\right)^2\right]^{-1}$$

is a geometric correction factor (Raviola et al., 2016)

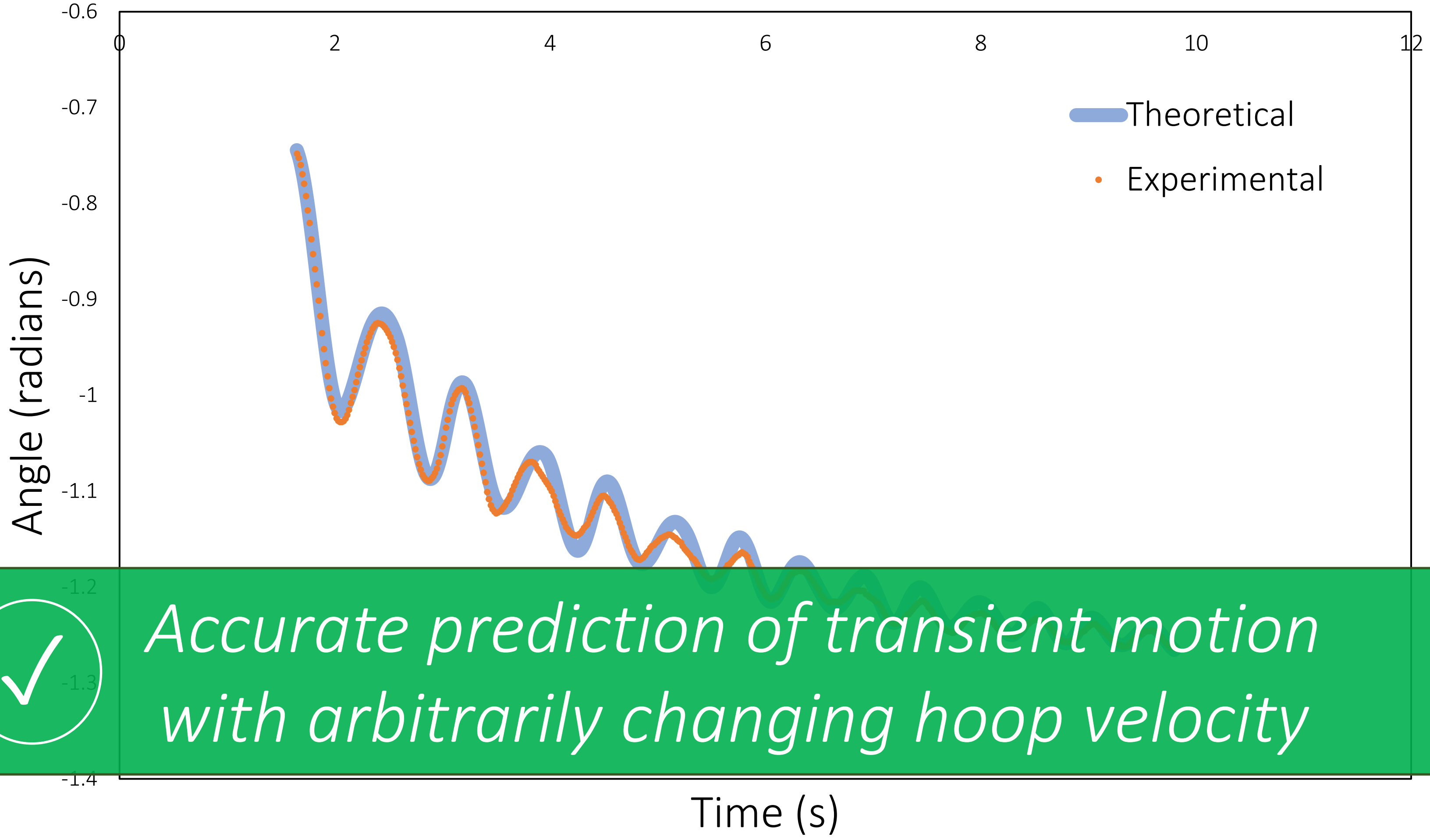
$$(\gamma=2/5)$$

Experimental Verification



Measure angular velocity and use as input into simulation to predict transient motion

Experimental Verification



✓ *Accurate prediction of transient motion with arbitrarily changing hoop velocity*

Steady State Solutions

$$\ddot{\theta} + \kappa\left[\frac{b}{m}\dot{\theta} + \left(\frac{g}{R_{CM}} - \omega^2\cos\theta\right)\sin\theta\right] = 0$$

$\theta_{eq} \Leftrightarrow \ddot{\theta} = \dot{\theta} = 0$ *Bead settles due to friction*

$$\big\downarrow$$

$$\sin\theta_{eq} = 0 \Longrightarrow \theta_{eq} = 0$$

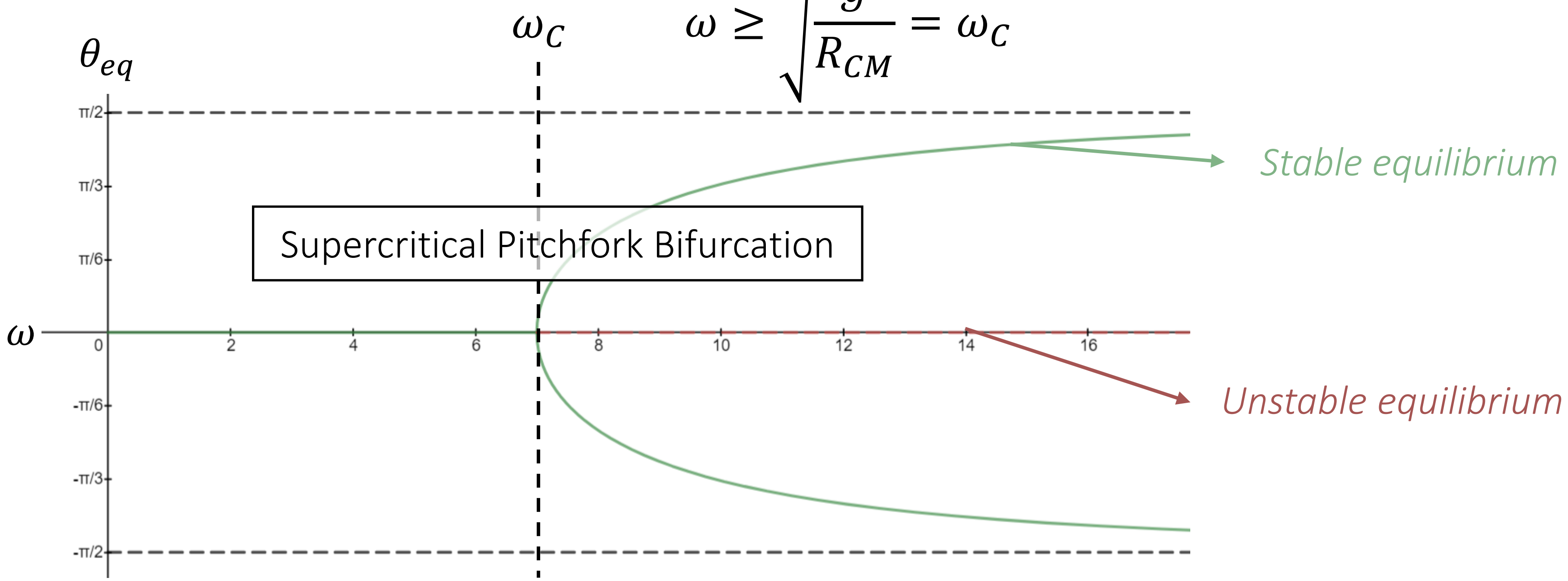
$$\cos\theta_{eq} = \frac{g}{\omega^2 R_{CM}} \Longrightarrow \theta_{eq} = \arccos\left(\frac{g}{\omega^2 R_{CM}}\right)$$

Examining Equilibria

$$\sin\theta_{eq} = 0 \implies \theta_{eq} = 0$$

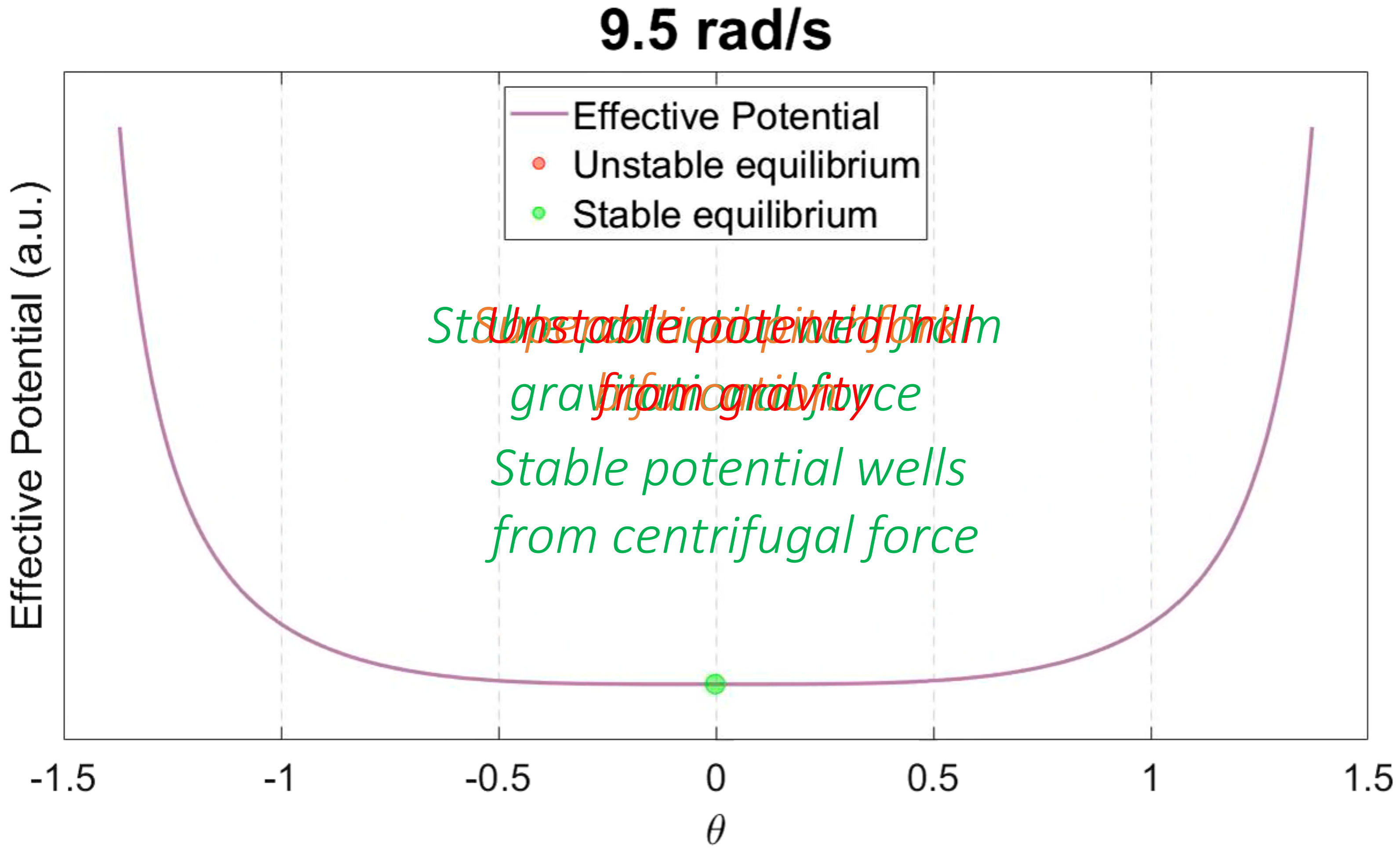
$$\cos\theta_{eq} = \frac{g}{\omega^2 R_{CM}} \implies \theta_{eq} = \arccos\left(\frac{g}{\omega^2 R_{CM}}\right)$$

$$\omega \geq \sqrt{\frac{g}{R_{CM}}} = \omega_C$$



Stability Analysis

$$U_{\text{eff}}(\theta) = U(\theta) + \frac{I^2\omega^2}{2mR_{CM}^2\cos^2\theta}$$



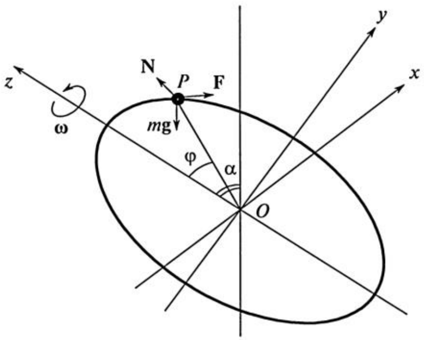
Stable equilibrium from gravitational force

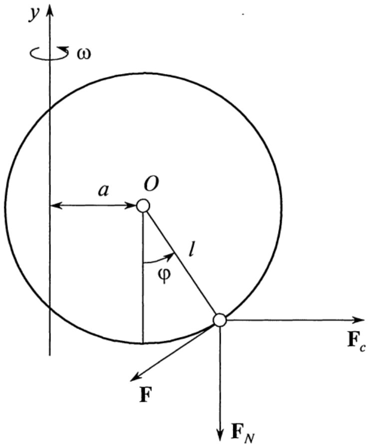
Supercritical pitchfork bifurcation

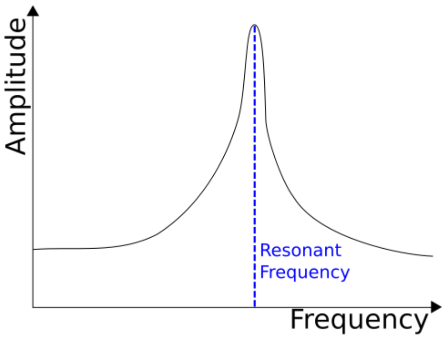
Unstable potential hill from gravity

Stable potential wells from centrifugal force

Theoretical Model





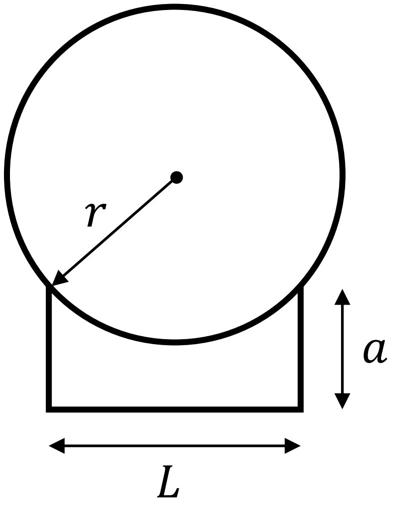


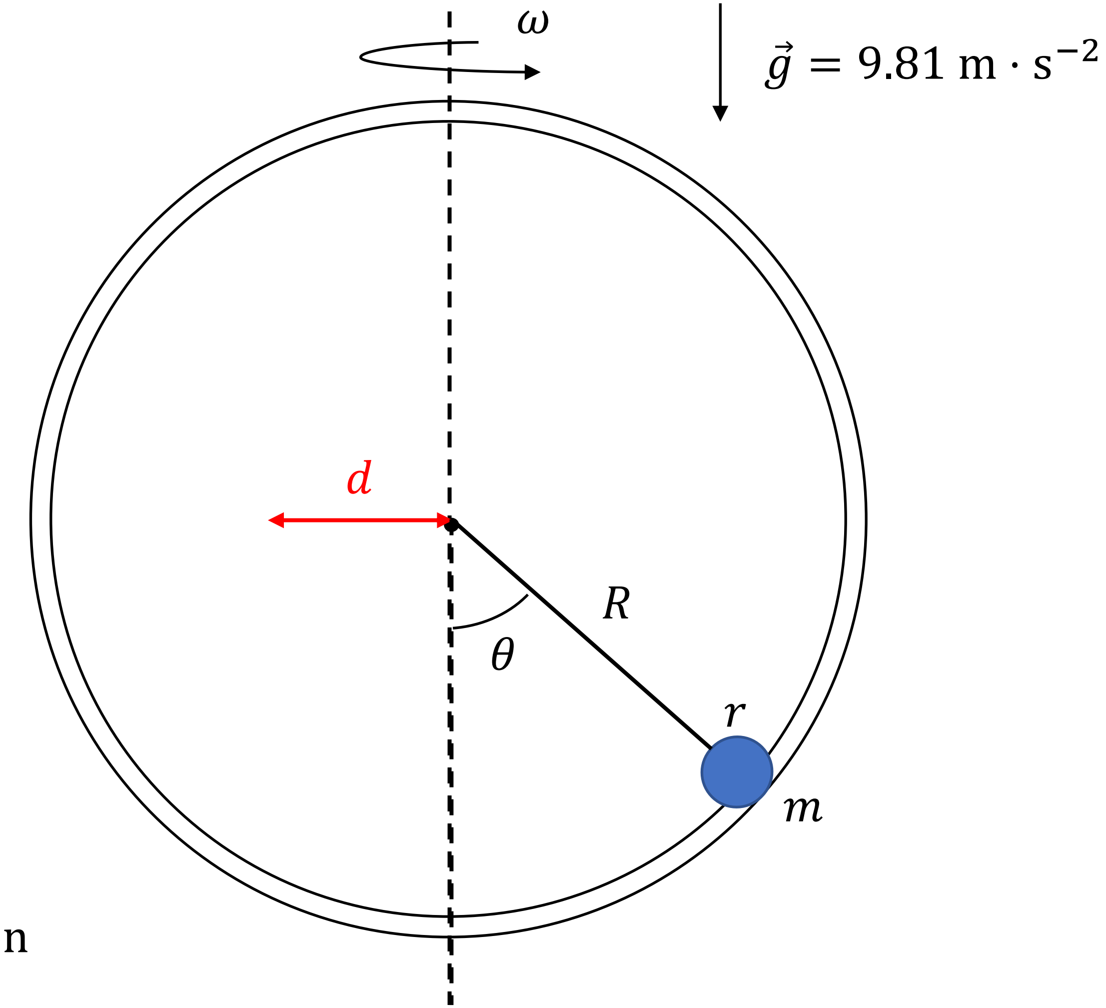
Lagrangian Analysis

Axis of Rotation

Inclination and Resonance

Geometry

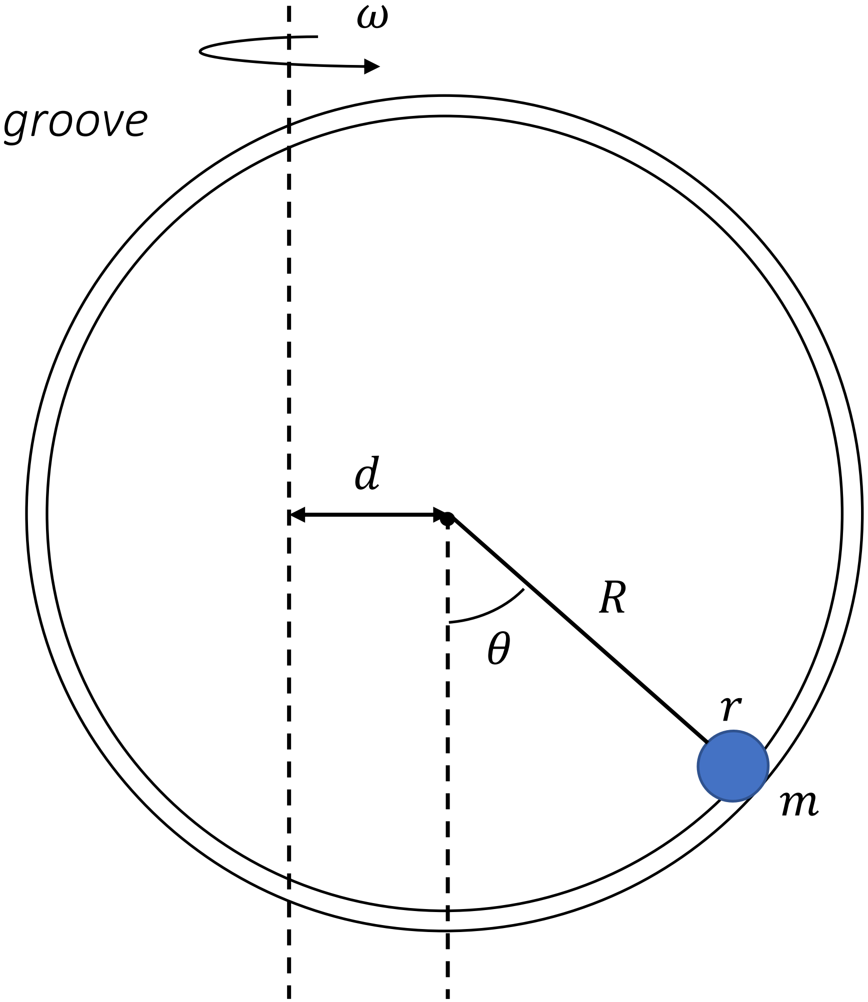




R : hoop radius
ω : hoop angular velocity
r : bead radius
m : bead mass
θ : bead inclination
g : gravitational acceleration
d : axis offset

Assumptions

- ✓ *Motion is constrained to along the groove*
  - ✓ *Restrict* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- ✓ *Air resistance is negligible*



Lagrangian Analysis

$$T = \frac{1}{2} m R_{CM}^2 \frac{\dot{\theta}^2}{\kappa} + \frac{1}{2} m \omega^2 (\underbrace{d}_{\text{Shifted axis}} + R_{CM} \sin\theta)^2$$

$$U = -mgR_{CM} \cos\theta$$

$$Q_\theta = -bR_{CM}^2 \dot{\theta}$$

where

$$\kappa = \left[1 + \gamma \frac{r^2}{R_{CM}^2} \left(\frac{R_{CM}}{\sqrt{r^2 - \frac{L^2}{4}}} + 1 \right)^2 \right]^{-1}$$

is a geometric correction factor (Raviola et al., 2016)

$$\boxed{(\gamma = 2/5)}$$

Coefficient of Moment of Inertia

Lagrangian Analysis

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)-\frac{\partial\mathcal{L}}{\partial\theta}$$

$$\ddot{\theta}+\kappa\left[\frac{b}{m}\dot{\theta}+\left(\frac{g}{R_{CM}}-\omega^2\cos\theta\right)\sin\theta-\frac{\omega^2 d\cos\theta}{R_{CM}}\right]=0$$

$$\theta_{eq}\Leftrightarrow\ddot{\theta}=\dot{\theta}=0$$

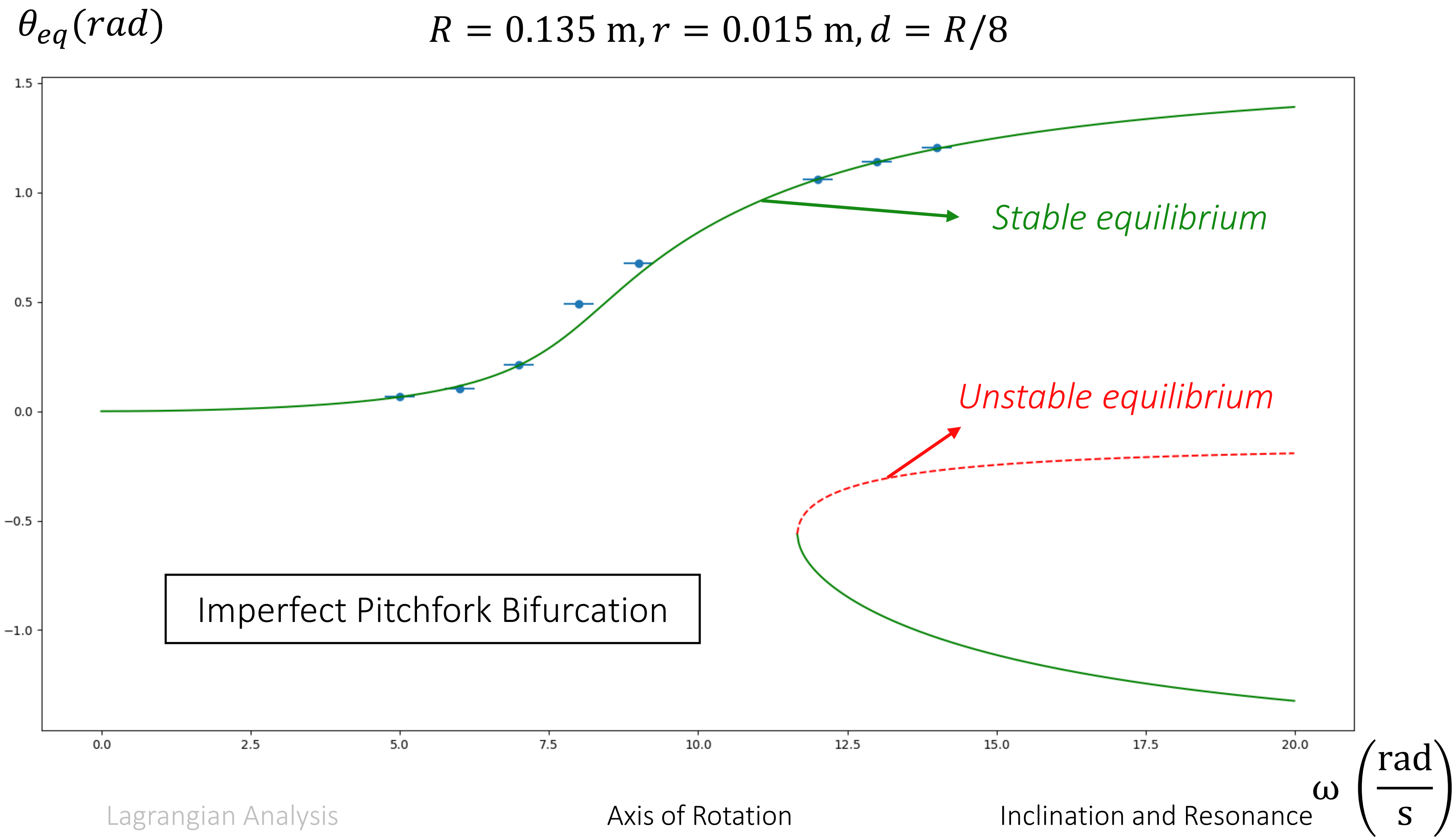
$$\left(\frac{g}{R_{CM}}-\omega^2\cos\theta\right)\sin\theta-\frac{\omega^2 d\cos\theta}{R_{CM}}=0$$

Equilibrium Solution

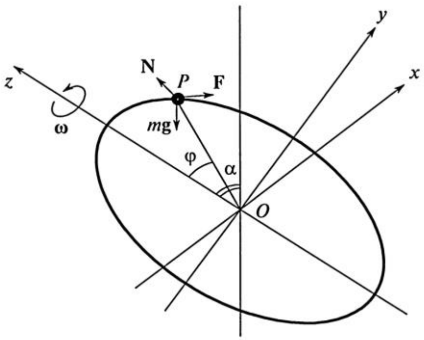
Python numerical solution with fsolve

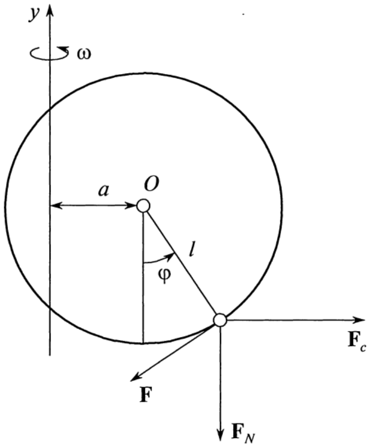
Examining Equilibria

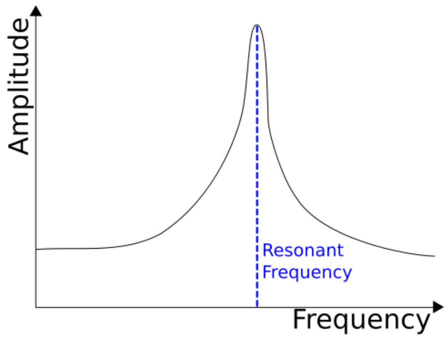
$R = 0.135\ \mathrm{m}, r = 0.015\ \mathrm{m}, d = R/8$



Theoretical Model





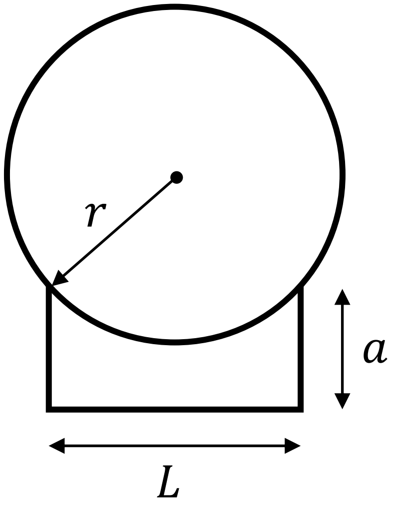


Lagrangian Analysis

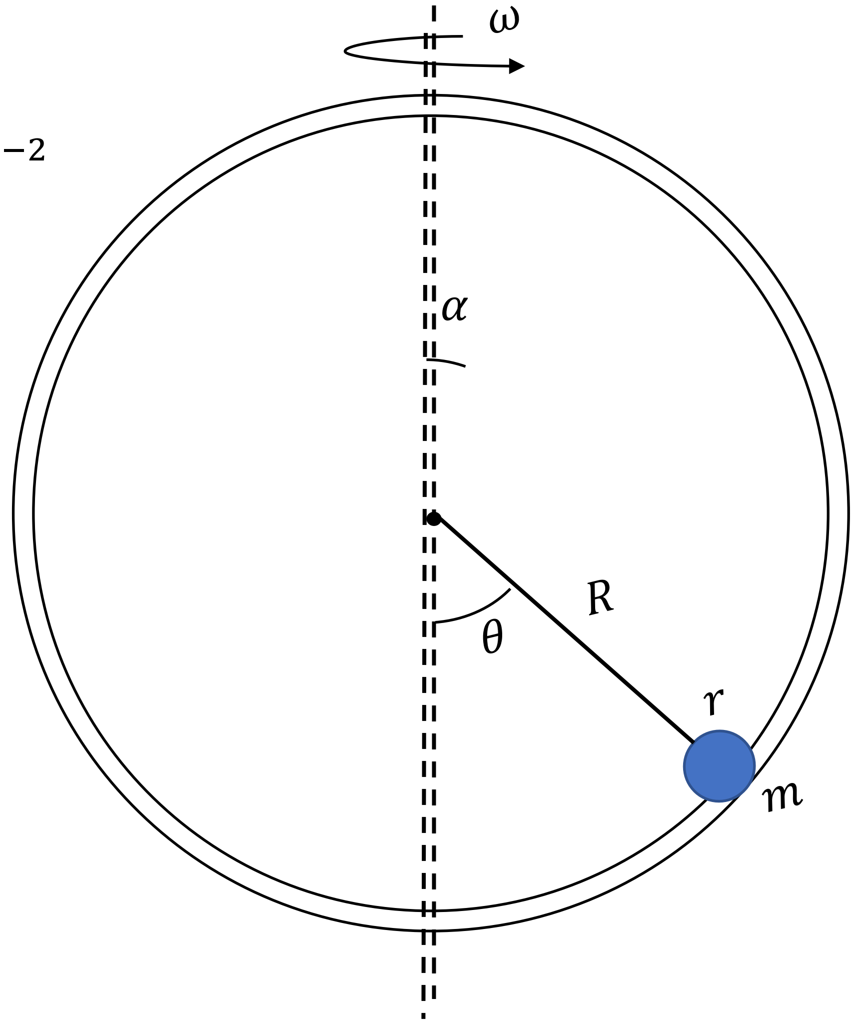
Axis of Rotation

Inclination and Resonance

Geometry

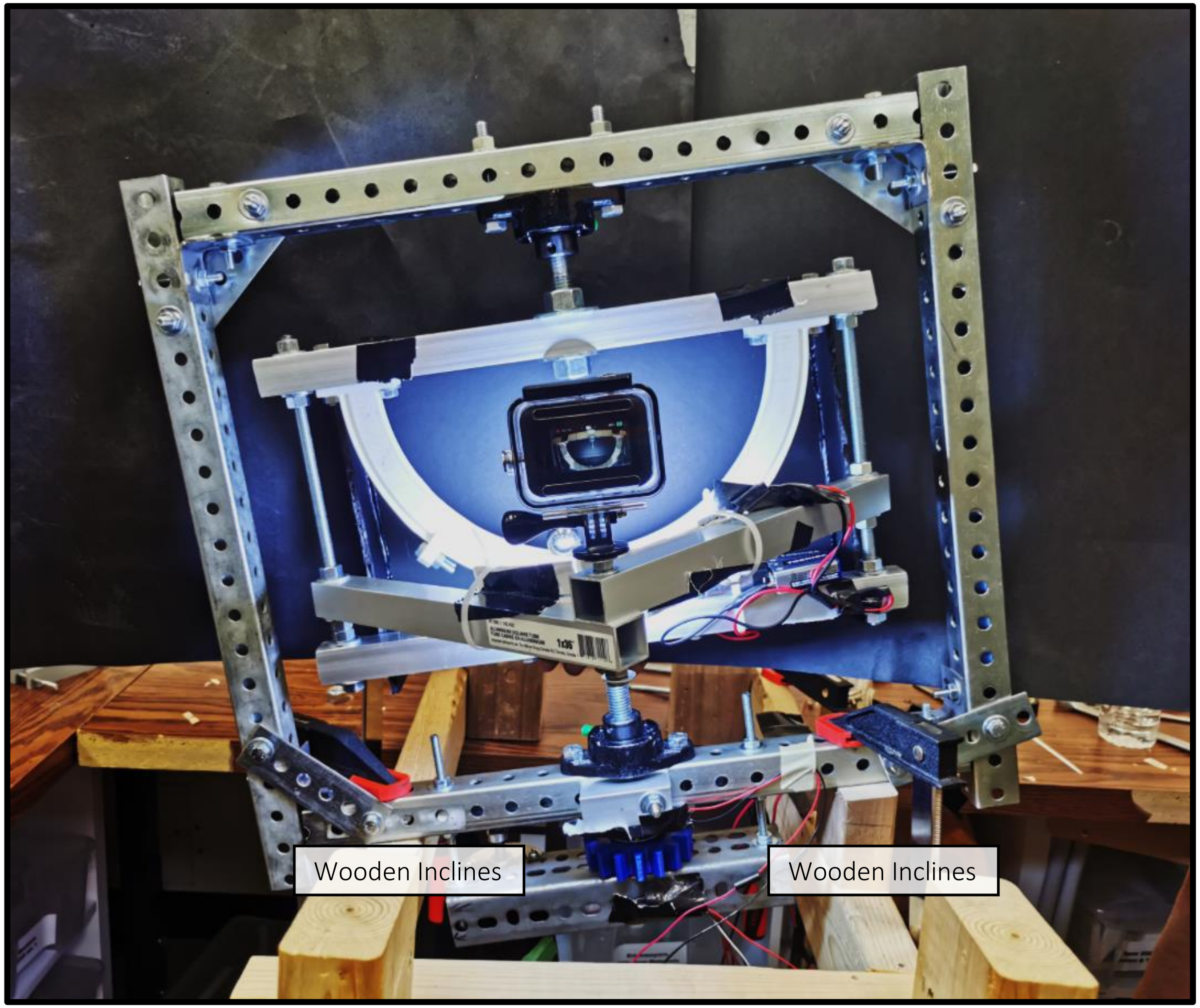


$\vec{g} = 9.81\ \mathrm{m \cdot s^{-2}}$



R = hoop radius
ω = hoop angular velocity
r = bead radius
m = bead mass
θ = bead inclination
g = gravitational acceleration
α = axis tilt

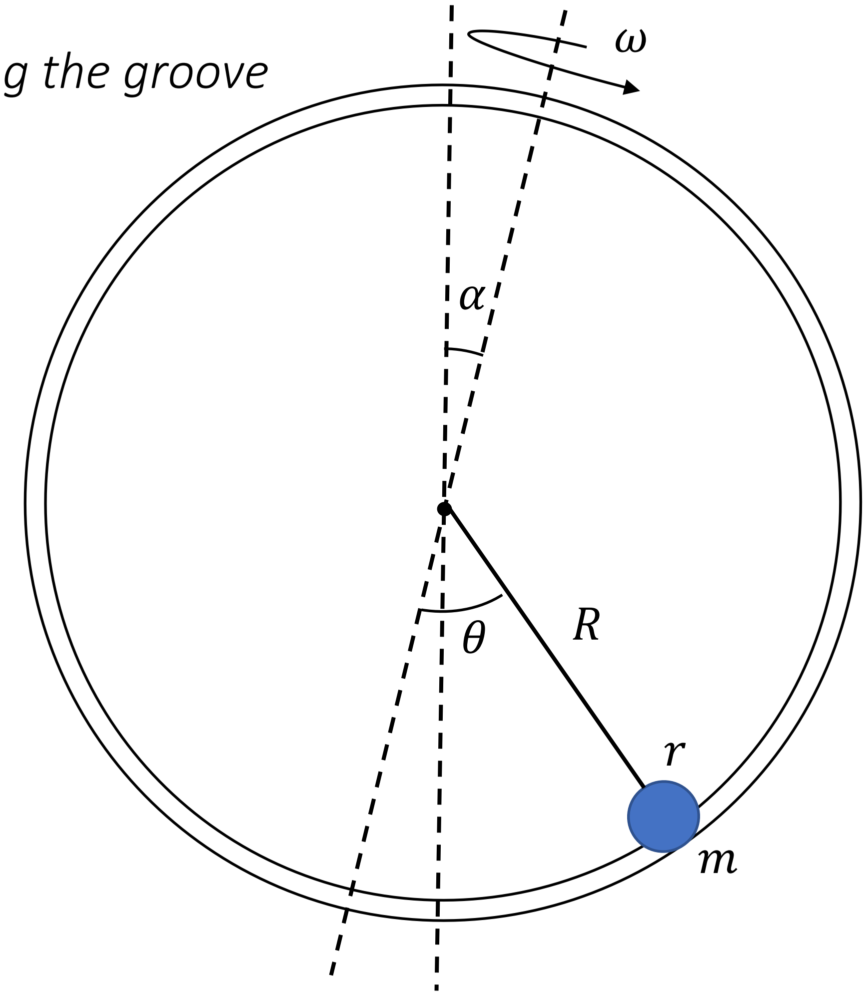
Experimental Setup



$$\alpha = 1.07° \pm 0.01°$$
$$\alpha = 2.75° \pm 0.01°$$
$$\alpha = 3.14° \pm 0.01°$$
$$\alpha = 4.65° \pm 0.01°$$
$$\alpha = 5.72° \pm 0.01°$$
$$\alpha = 10.25° \pm 0.01°$$

Assumptions

- ✓ *Motion is constrained to along the groove*
- ✓ *Air resistance is negligible*



Lagrangian Analysis

$$T = \frac{1}{2} m R_{CM}^2 \left(\frac{\dot{\theta}^2}{\kappa} + \omega^2 \sin^2 \theta \right)$$

$$U = -mgR_{CM}(\cos\alpha\cos\theta + \sin\alpha\cos(\omega t)\sin\theta)$$

$$Q_\theta = -bR_{CM}^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta}$$

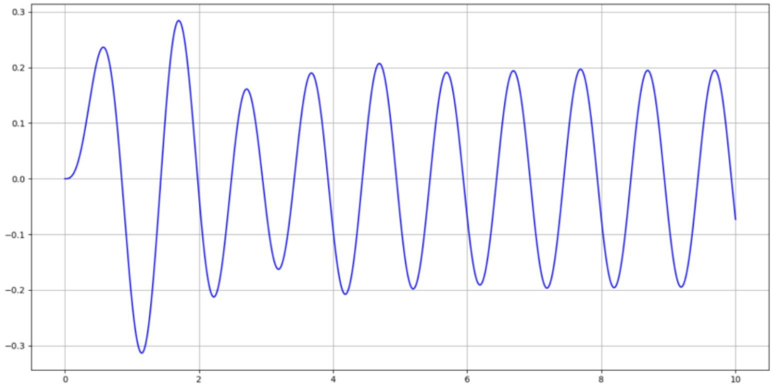
$$\ddot{\theta} + \kappa\left[\frac{b}{m}\dot{\theta} + \left(\frac{g}{R_{CM}}\cos\alpha - \omega^2\cos\theta\right)\sin\theta\right] = \kappa\frac{g}{R_{CM}}\sin\alpha\cos(\omega t)\cos\theta$$

Oscillation and Resonance

$$\ddot{\theta} + \kappa\left[\frac{b}{m}\dot{\theta} + \left(\frac{g}{R_{CM}}\cos\alpha - \omega^2\cos\theta\right)\sin\theta\right] = \kappa\frac{g}{R_{CM}}\sin\alpha\cos(\omega t)\cos\theta$$

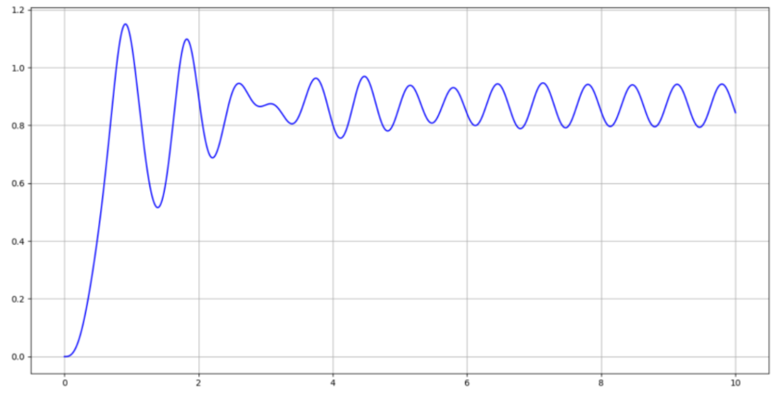
Solve with scipy odeint

$\theta(rad)$



$\omega = 2\pi$

$t(s)$



$\omega = 3\pi$

$t(s)$

$b = 0.09\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$
$m = 0.066\ \mathrm{kg}$
$R = 0.183\ \mathrm{m}$
$r = 0.0126\ \mathrm{m}$
$L = 0.0174\ \mathrm{m}$
$\alpha = 4.5^\circ$
$\theta_0 = \dot{\theta}_0 = 0$

Small Angle Approximation

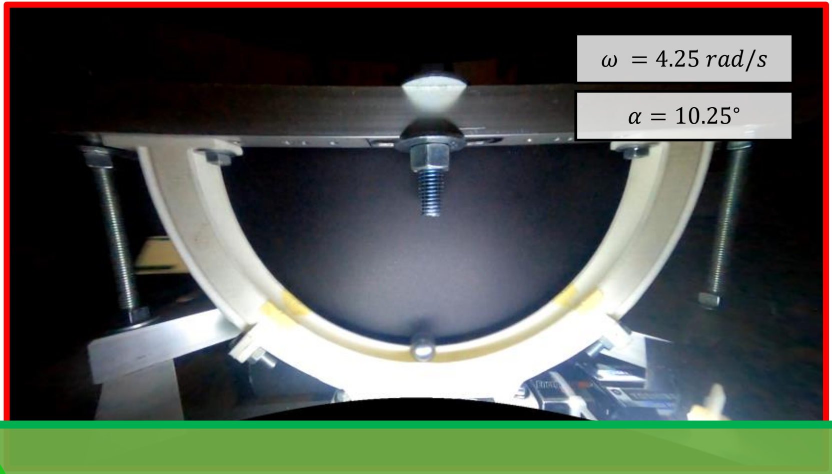
For Small Angles less than 15 degrees, simplify via Small Angle Approx.

$$\ddot{\theta} = k\alpha \frac{g}{R_{CM}} \underbrace{\cos(\omega t)}_{\boxed{\textit{Periodic Driving Term}}} - k\left[\frac{b}{m}\dot{\theta} + \left(\frac{g}{R_{CM}} - \omega^2\right)\theta\right]$$

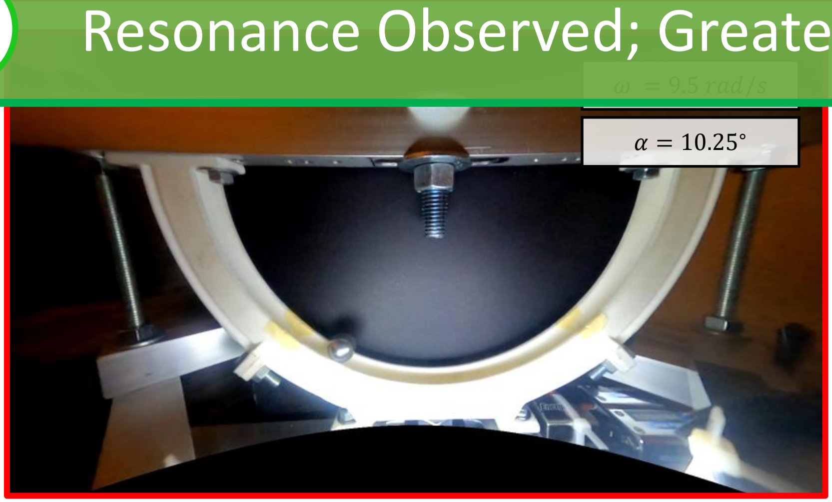
✓ *Periodic Variation of Cosine Causes Resonance in System*

(Raviola, L. A., Véliz, M. E., Salomone, H. D., Olivieri, N. A., & Rodríguez, E. E. 2016)

Experimental Observation



✓ Resonance Observed; Greatest Amplitude at around 5 Rad/S



Resonance

Maximum Amplitude Achieved:

$$A_s(\omega) = \frac{\boxed{\alpha g}}{R_{CM}\sqrt{\left(\left(\frac{k+1}{k}\right)\omega^2 - \boxed{\frac{g}{R_{CM}}}\right)^2 + \boxed{\left(\frac{\beta}{m}\right)^2 \omega^2}}}$$

Driving Term

Natural Frequency

Damping Term

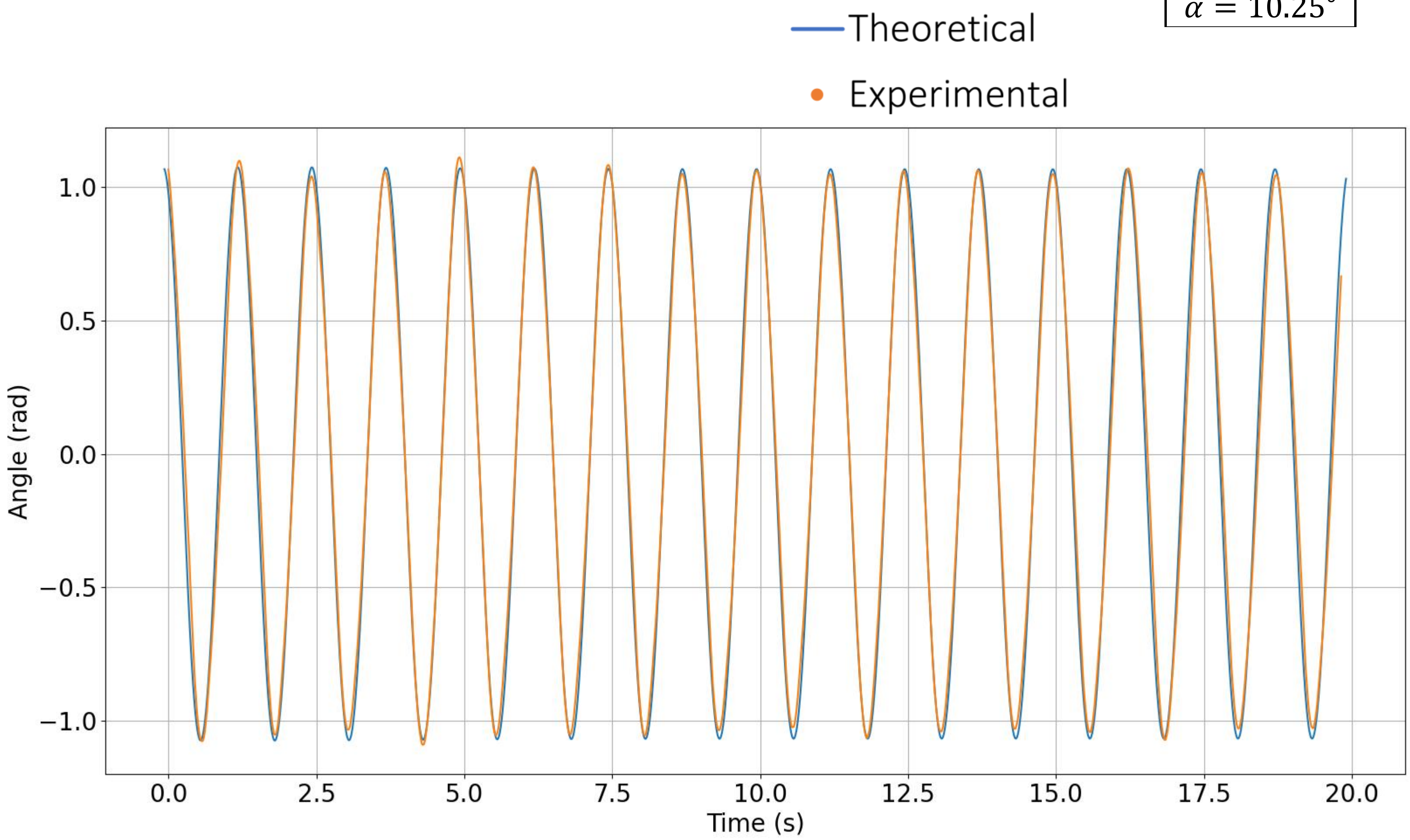
Resonant Frequency:

$$\omega_S^{res} = \sqrt{\frac{k+1}{k}\left(\frac{g}{R_{CM}} - \frac{k}{2(k+1)}\frac{b^2}{m^2}\right)}$$

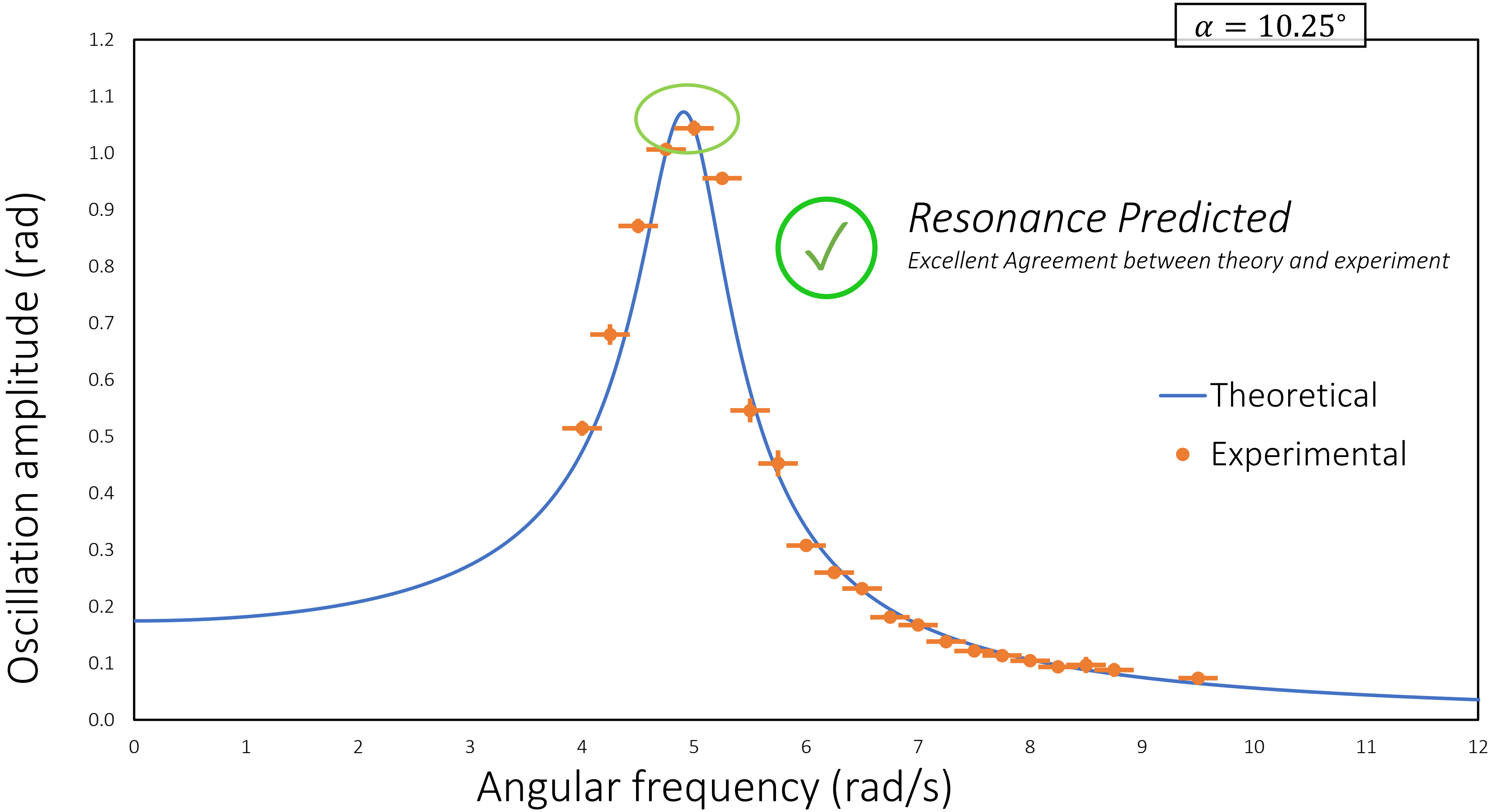
(Raviola, L. A., Véliz, M. E., Salomone, H. D., Olivieri, N. A., & Rodríguez, E. E. 2016)

Resonance Steady-State

$\alpha = 10.25^\circ$



Resonance



Resonance Predicted

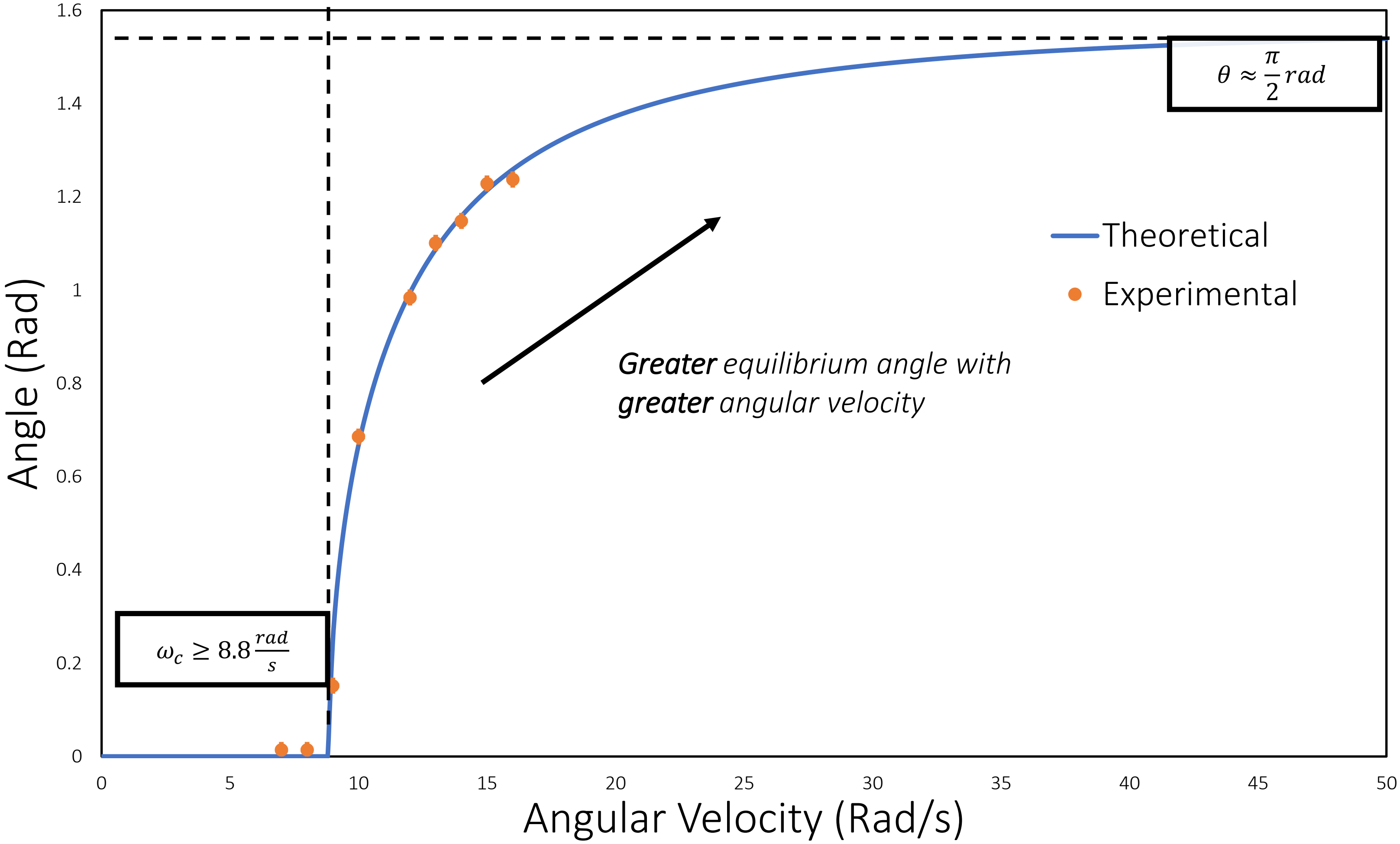
Excellent Agreement between theory and experiment

Raviola, L. A., Véliz, M. E., Salomone, H. D., Olivieri, N. A., & Rodríguez, E. E. (2016). The bead on a rotating hoop revisited: an unexpected resonance. European Journal of Physics, 38(1), 015005. Doi.org/10.1088/0143-0807/38/1/015005

Key Parameters

Varying Angular Velocity

Radius= 0.127 m
M = 0.0066 kg



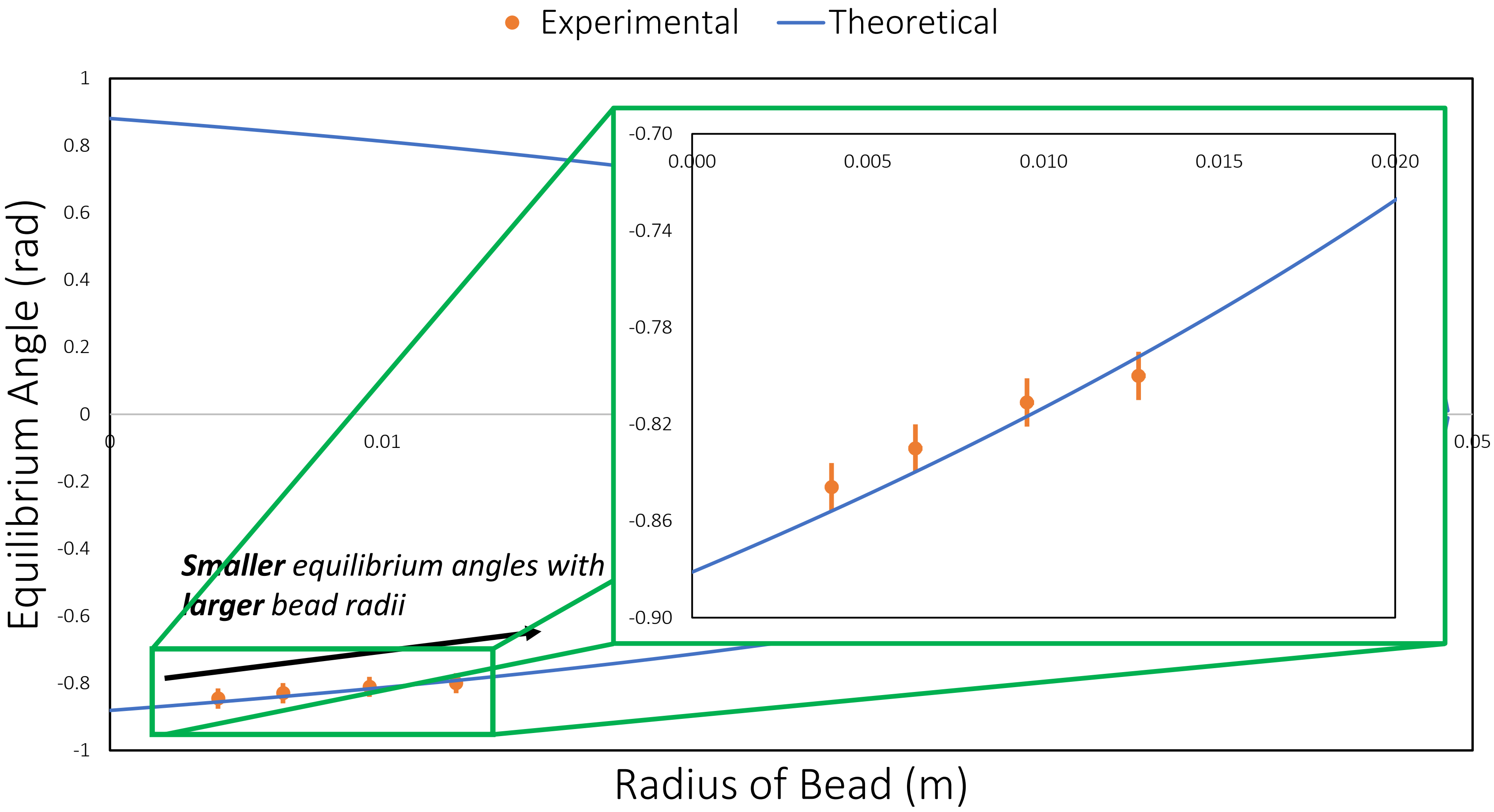
$\theta \approx \frac{\pi}{2} rad$

$\omega_c \geq 8.8 \frac{rad}{s}$

Greater *equilibrium angle with* ***greater*** *angular velocity*

Varying Bead Radius

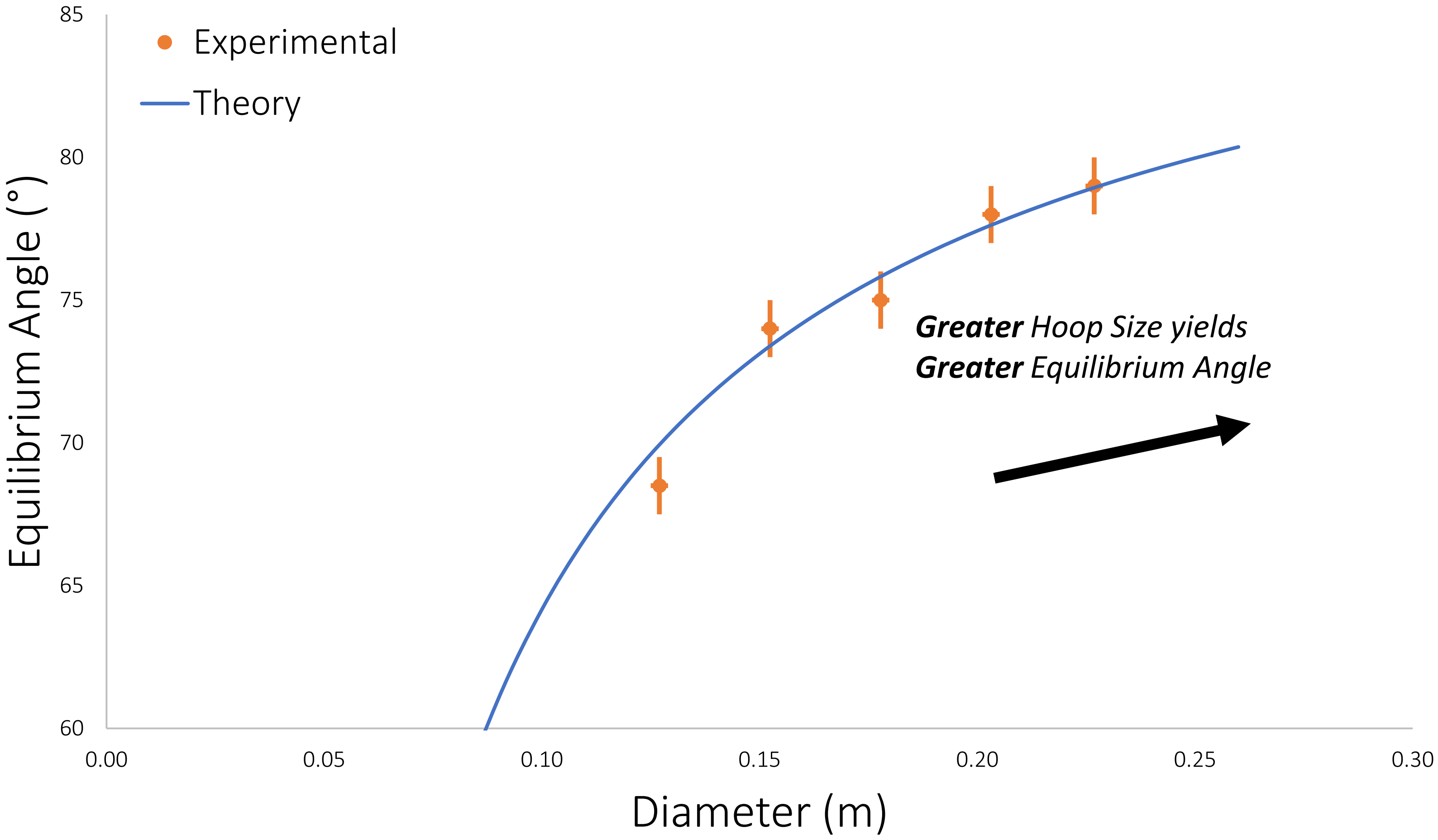
$$R = 0.135\ \mathrm{m}$$
$$\omega = 10.68\ \mathrm{rad \cdot s^{-1}}$$
$$g = 9.8\ \mathrm{m \cdot s^{-2}}$$



***Smaller** equilibrium angles with **larger** bead radii*

Varying Hoop Diameter

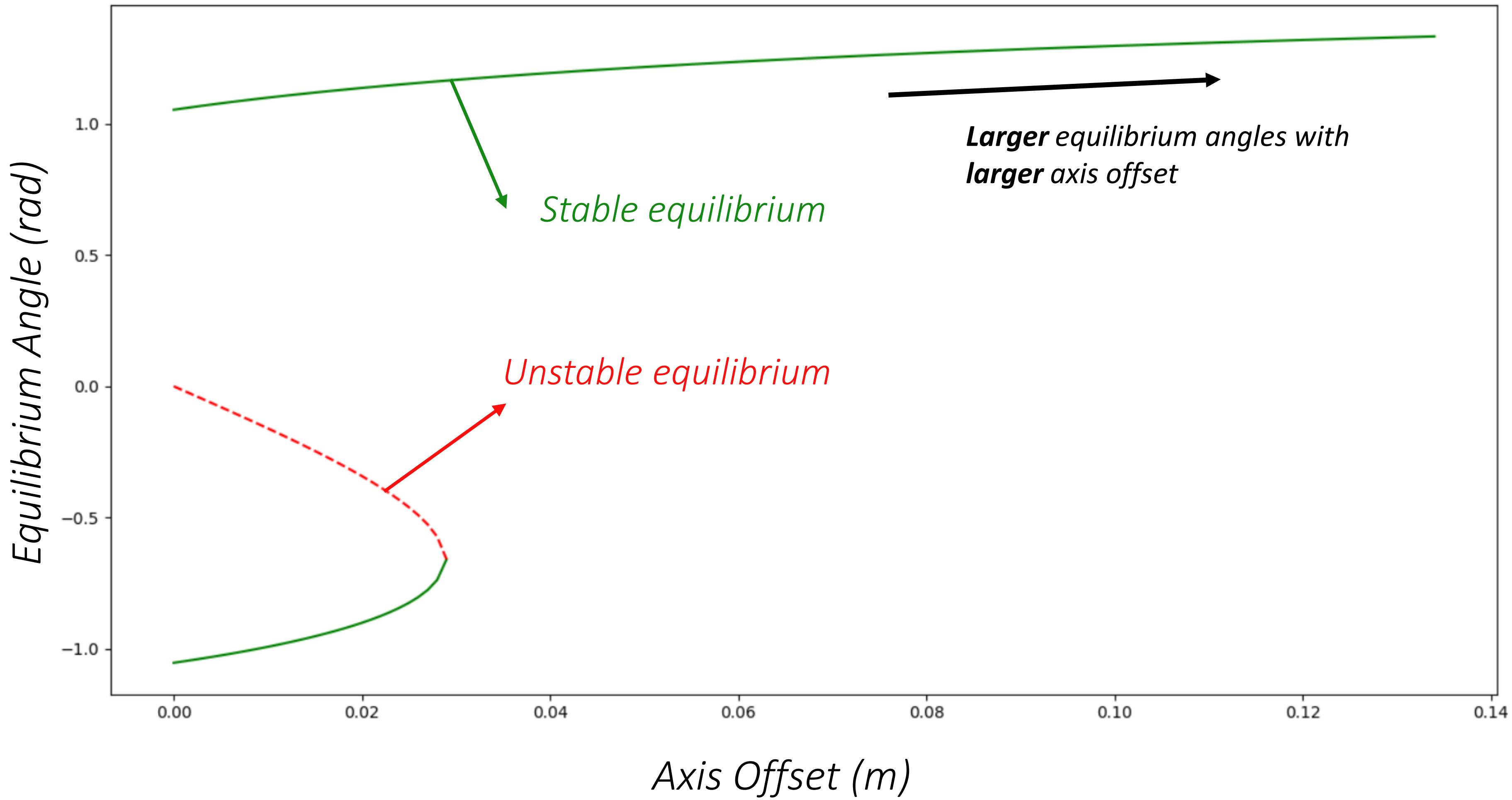
M = 0.002 kg



Greater *Hoop Size yields*
Greater *Equilibrium Angle*

Varying Axis Offset

$$R = 0.135\ \mathrm{m}$$
$$r = 0.0095\ \mathrm{m}$$
$$\omega = 12.57\ \mathrm{rad} \cdot \mathrm{s}^{-1}$$
$$g = 9.8\ \mathrm{m} \cdot \mathrm{s}^{-2}$$



Larger *equilibrium angles with* ***larger*** *axis offset*

Points Raised

Qualitative Explanation of Phenomenon (Opponent)

- *Clarified the role of the centrifugal force in the onset of the phenomenon*

Axial offset and tilt (Reviewer)

- *Important to investigate, within the scope of the problem*
- *Hoop is still rotating vertically with axial offset*
- *Tilt is rotating about a diameter*

Drag, Rolling Friction (Opponent)

- *Independently identified friction, considered normal force*
- *Verified rolling without slipping*

Dynamics of the system (Opponent and Reviewer)

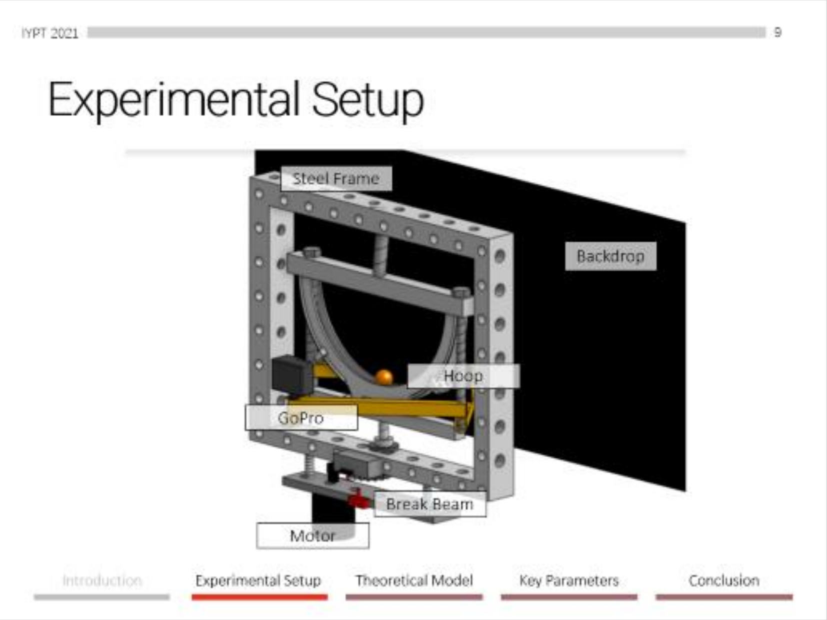
- ***Arbitrary** angular velocity input, resonance conditions, great agreement*
- *Static measurements are also important for the phenomenon*

Parameter Variation (Reporter/Opponent/Reporter and Opponent)

- *Hoop radius, angular velocity, bead radius, axial offset, tilt varied – sufficient variation*

Conclusion

A circular hoop rotates about a vertical diameter. A small bead is allowed to roll in a groove on the inside of the hoop. Investigate the relevant parameters affecting the dynamics of the bead.



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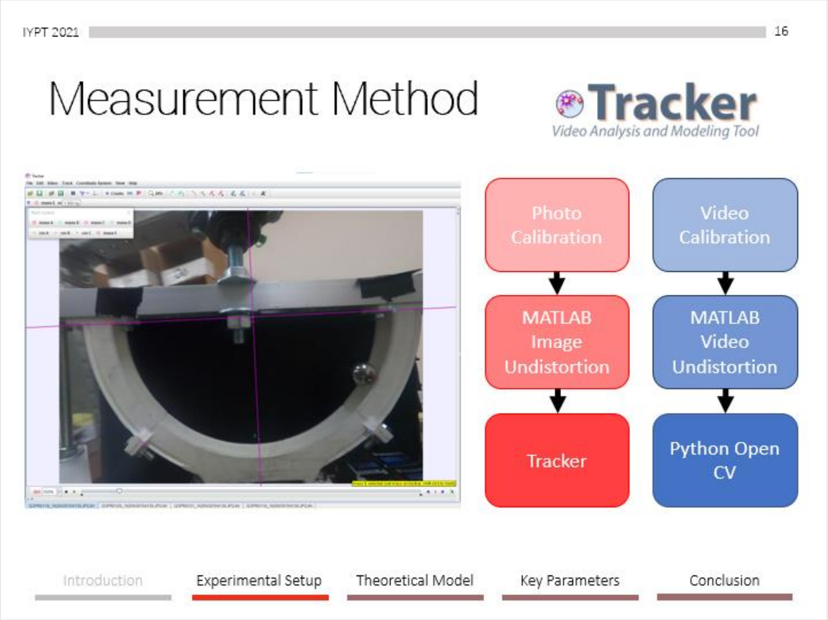
Experimental Setup

Introduction Experimental Setup Theoretical Model Key Parameters Conclusion



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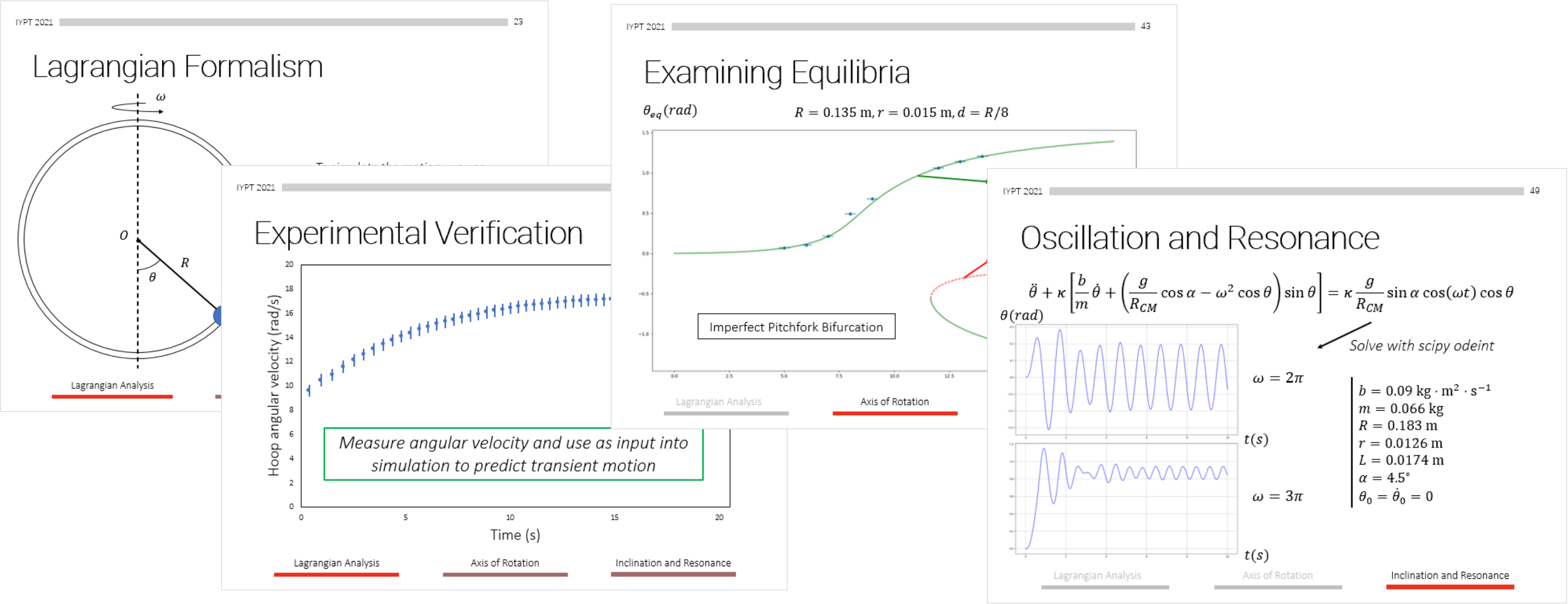
Measurement Method

Tracker
Video Analysis and Modeling Tool

Introduction Experimental Setup Theoretical Model Key Parameters Conclusion

Conclusion

A ***circular hoop rotates*** *about a vertical diameter. A small bead is allowed to* ***roll in a groove*** *on the inside of the hoop. Investigate the* ***relevant parameters*** *affecting the* ***dynamics of the bead****.*



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Lagrangian Formalism

IYPT 2021

Experimental Verification

Measure angular velocity and use as input into simulation to predict transient motion

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Examining Equilibria

$\theta_{eq}(rad)$ $R = 0.135$ m, $r = 0.015$ m, $d = R/8$

Imperfect Pitchfork Bifurcation

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Oscillation and Resonance

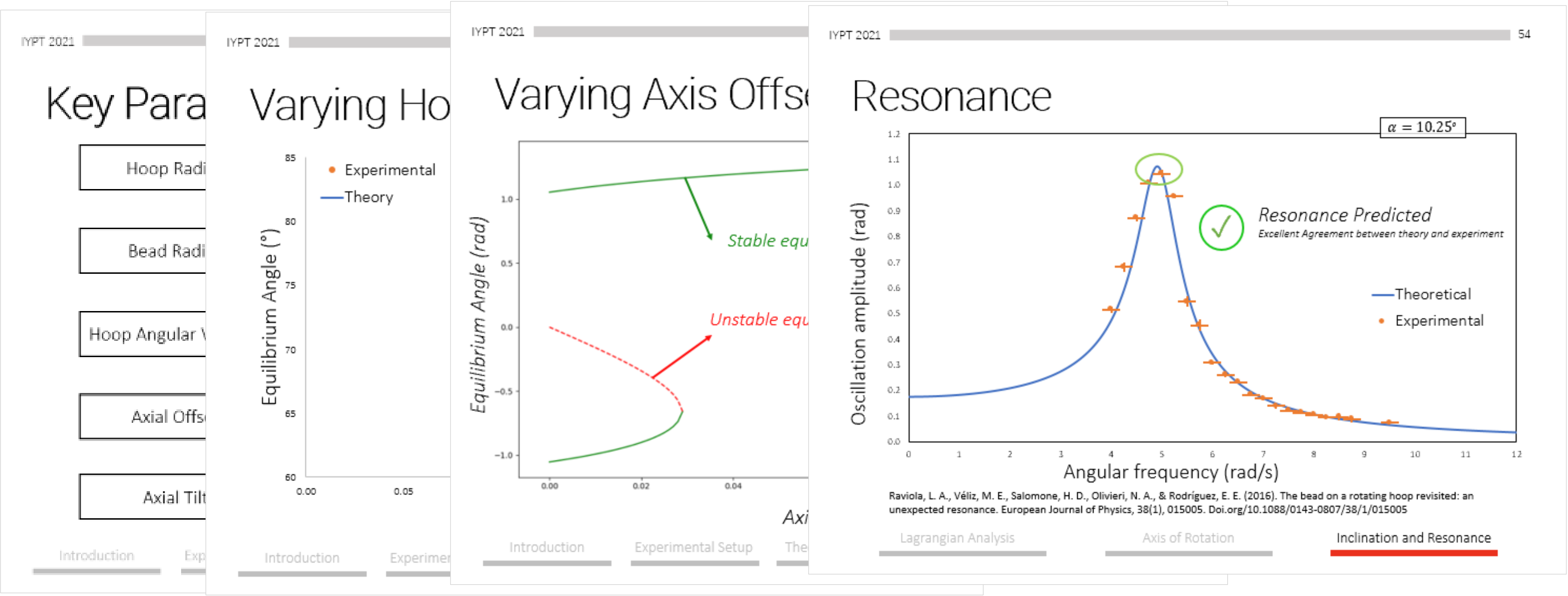
$$\ddot{\theta} + \kappa\left[\frac{b}{m}\dot{\theta} + \left(\frac{g}{R_{CM}}\cos\alpha - \omega^2\cos\theta\right)\sin\theta\right] = \kappa\frac{g}{R_{CM}}\sin\alpha\cos(\omega t)\cos\theta$$

Solve with scipy odeint

$b = 0.09\ \mathrm{kg\cdot m^2\cdot s^{-1}}$
$m = 0.066$ kg
$R = 0.183$ m
$r = 0.0126$ m
$L = 0.0174$ m
$\alpha = 4.5°$
$\theta_0 = \dot{\theta}_0 = 0$

Conclusion

*A **circular hoop rotates** about a vertical diameter. A small bead is allowed to **roll in a groove** on the inside of the hoop. Investigate the **relevant parameters** affecting the **dynamics of the bead**.*



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References

Balandin, D., & Shalimova, E. (2015). Bifurcations of the relative equilibria of a heavy bead on a hoop uniformly rotating about an inclined axis with DRY FRICTION. Journal of Applied Mathematics and Mechanics, 79(5), 440-445. doi:10.1016/j.jappmathmech.2016.03.004

Burov, A., & Yakushev, I. (2014). Bifurcations of the relative equilibria of a heavy bead on a Rotating hoop with DRY FRICTION. Journal of Applied Mathematics and Mechanics, 78(5), 460-467. doi:10.1016/j.jappmathmech.2015.03.004

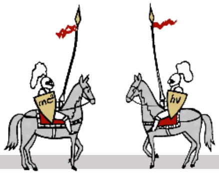
Cross, R. (2016). Coulomb's law for rolling friction. American Journal of Physics, 84(3), 221-230. doi:10.1119/1.4938149

References

Mancuso, R. V. (2000). A working mechanical model For first- and Second-order phase transitions and the cusp catastrophe. American Journal of Physics, 68(3), 271-277. doi:10.1119/1.19403

Raviola, L. A., Véliz, M. E., Salomone, H. D., Olivieri, N. A., & Rodríguez, E. E. (2016). The bead on a rotating hoop revisited: An unexpected resonance. European Journal of Physics, 38(1), 015005. doi:10.1088/0143-0807/38/1/015005

Thank you for listening



Appendix

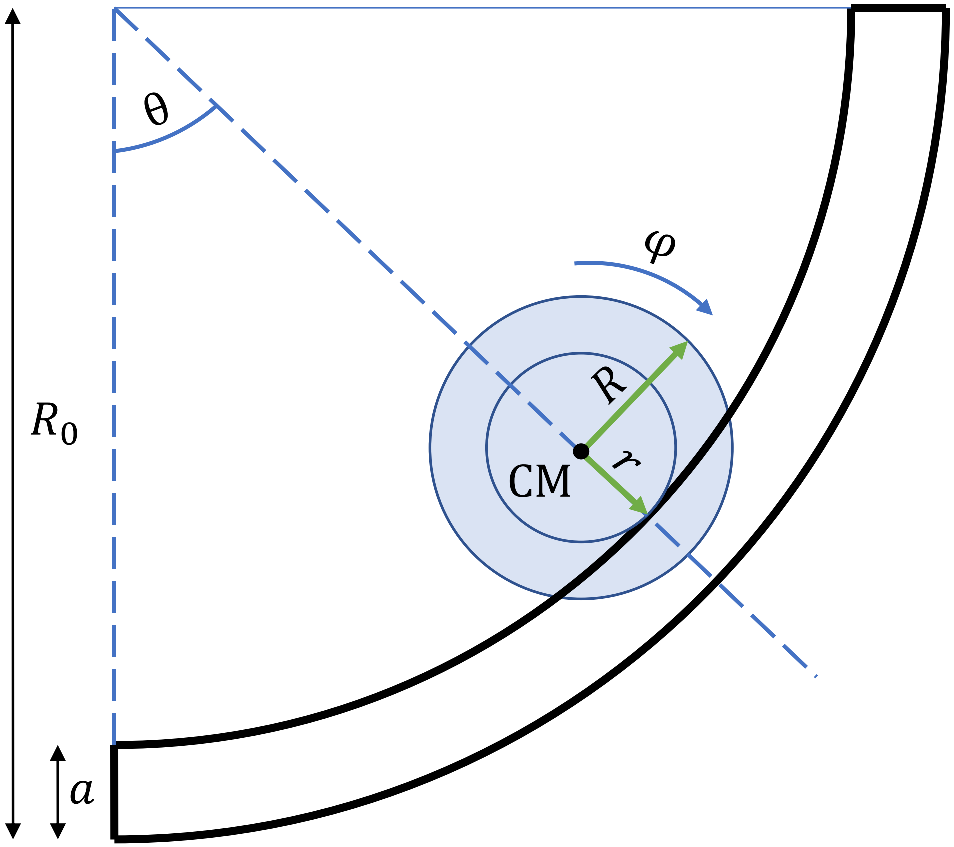
Appendix Static Friction

$$I_{ball}\ddot{\varphi} = F_f r$$

$$r\dot{\varphi} = R_{CM}\dot{\theta}$$

$$F_f = \frac{2}{5} m R_{ball}^2 \frac{R_{CM}}{r^2} \ddot{\theta}$$

Appendix



Raviola et al., 2016.

Appendix

$$\mathrm{d}s = r\,\mathrm{d}\varphi = (R_0 - a)\,\mathrm{d}\theta$$

$$\dot{\varphi} = \frac{(R_0 - a)}{r}\dot{\theta}$$

$$\mathrm{d}s_{CM} = (R_0 - a - r)\,\mathrm{d}\theta = R_{CM}\,\mathrm{d}\theta$$

$$v_{CM,rel} = R_{CM}\dot{\theta}$$

Raviola et al., 2016.

Appendix

$$T = \frac{1}{2} m\left(v_{CM,rel}^2 + (R_{CM} \sin\theta\omega)^2\right) + \frac{1}{2} I_{CM} \dot{\varphi}^2$$

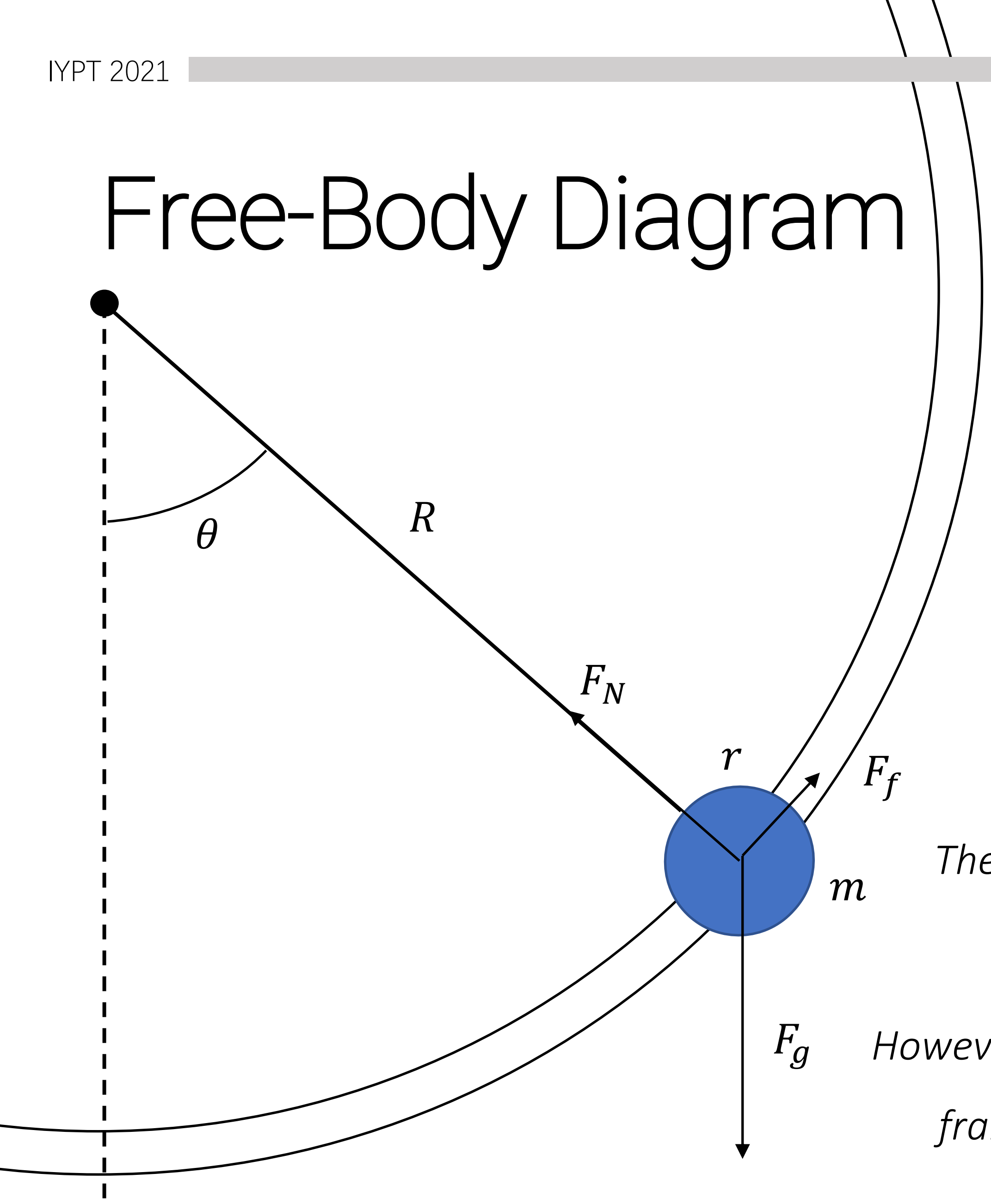
$$= \frac{1}{2} m R_{CM}^2 \left(\dot{\theta} + \omega^2 \sin^2\theta\right) + \frac{1}{2} m\gamma \frac{R^2}{r^2} (R_0 - a)^2\, \dot{\theta}^2$$

$$= \frac{1}{2} m R_{CM}^2 \left[\left(1 + \gamma \frac{R^2 (R_0 - a)^2}{R_{CM}^2 r^2}\right) \dot{\theta}^2 + \omega^2 \sin^2\theta\right]$$

$$= \frac{1}{2} m R_{CM}^2 \left(\frac{1}{\kappa} \dot{\theta}^2 + \omega^2 \sin^2\theta\right)$$

Raviola et al., 2016.

Free-Body Diagram

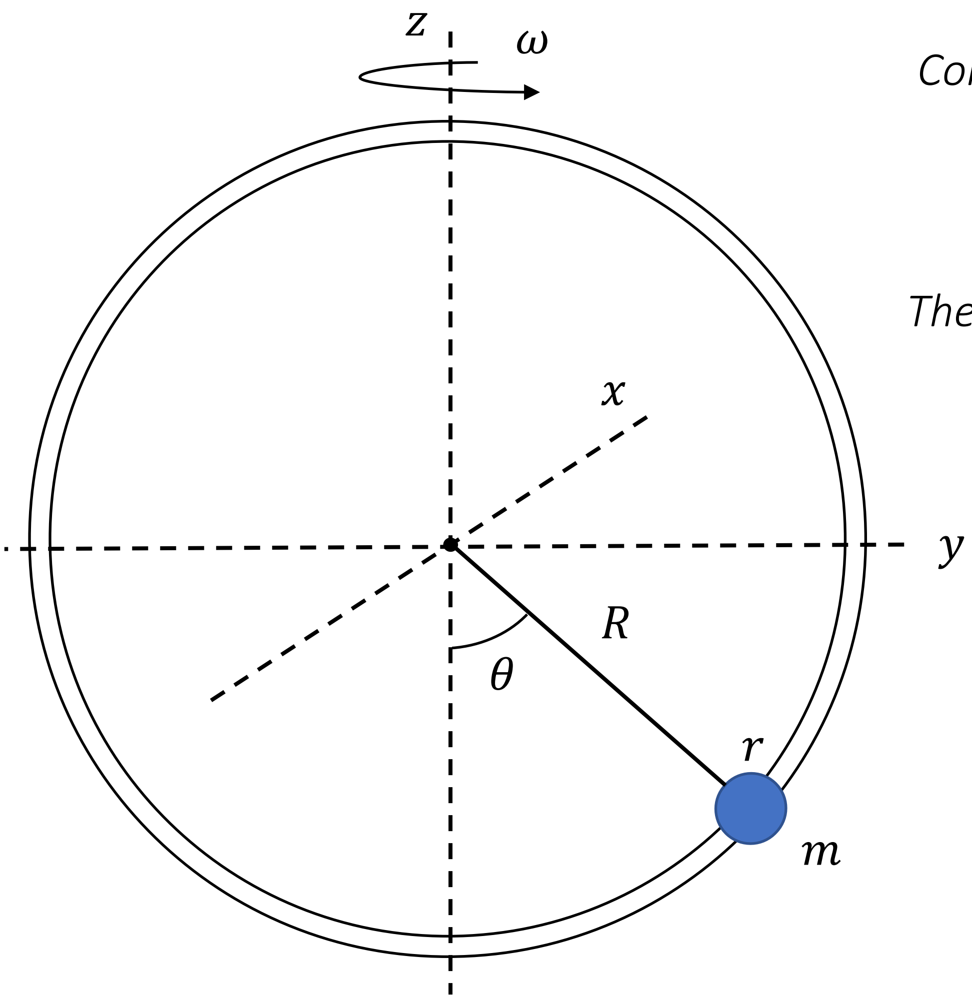


There are three forces acting on the bead

↓

However, we now shift into a rotating reference frame, which introduces fictitious forces

Motion Constraints



Consider a moving coordinate system that rotates along with the hoop

The constraints on the motion of the bead are:

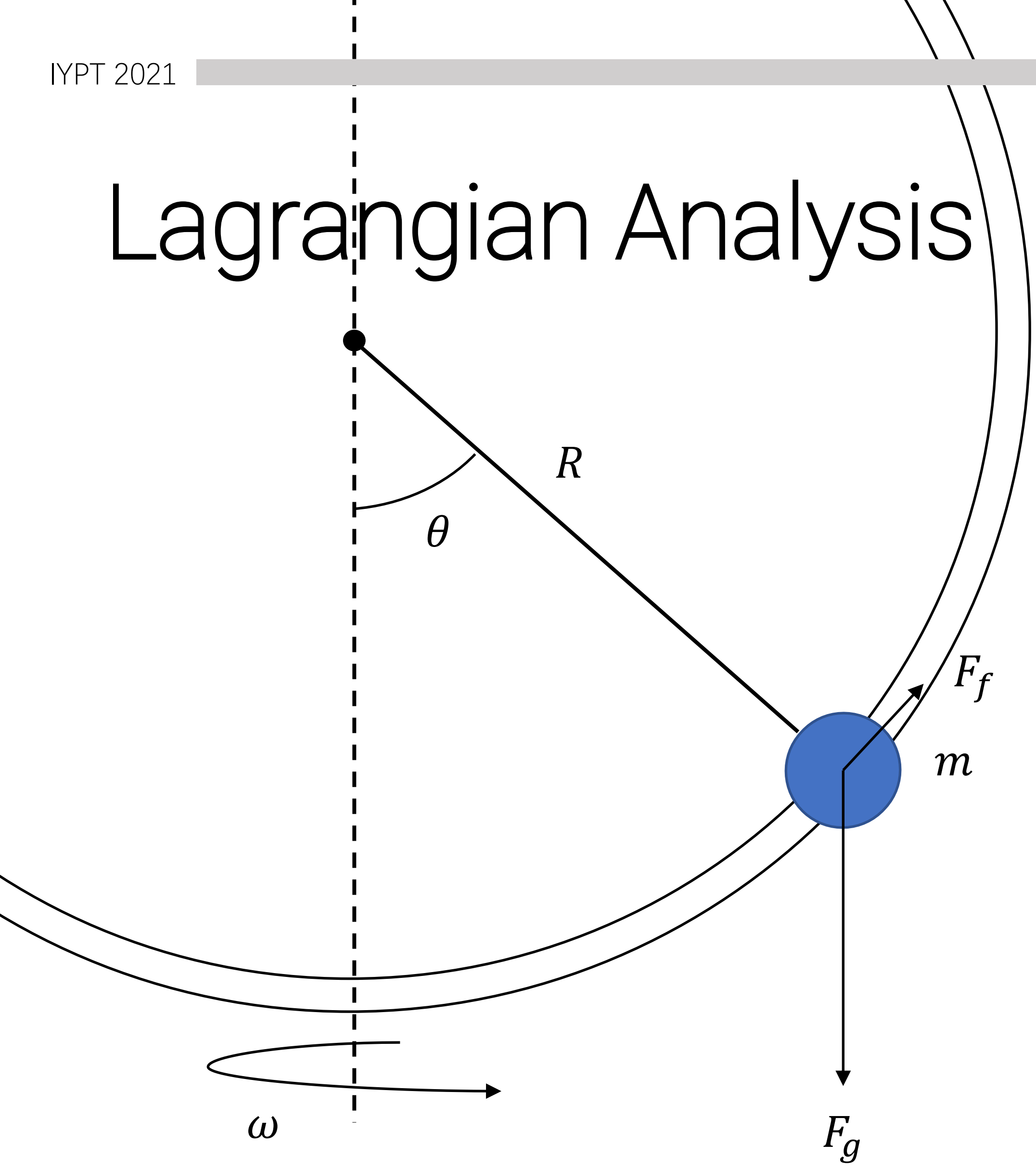
$$y^2 + z^2 = R^2$$

$$x = 0$$

Writing the force equation, we get

$$ma = F_{Cor} + F_c + F_g + N + F$$

Lagrangian Analysis



Note: $F_{Euler} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r} = 0$

Writing the force equation, we get

$$ma = F_C + F_c + F_g + F_N + F_f$$

Where

$$F_{Cor} = \begin{pmatrix} 0 \\ -2m\omega\dot{y} \\ 2m\omega\dot{x} \end{pmatrix}$$ *Coriolis Force*

$$F_c = \begin{pmatrix} 0 \\ m\omega^2 y \\ m\omega^2 z \end{pmatrix}$$ *Centrifugal Force*

$$F_g = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$ *Gravity*

$$F_f = -\mu \frac{\boldsymbol{v_r}}{v_r} N$$ *Sliding Friction*

Combined Solution

If the bead is in equilibrium, its relative velocity (to the hoop) and the Coriolis force will be equal to zero. Then, we can get that

$$\left|F_f\right| \leq \mu m g \left|\frac{z}{R} - \frac{\omega^2 y^2}{Rg}\right|$$

$$\frac{y}{l}\left(\frac{F_f y}{mgR} - 1\right) - \frac{z}{l}\left(\frac{\omega^2 y}{g} - \frac{F_f z}{mgR}\right) = 0$$

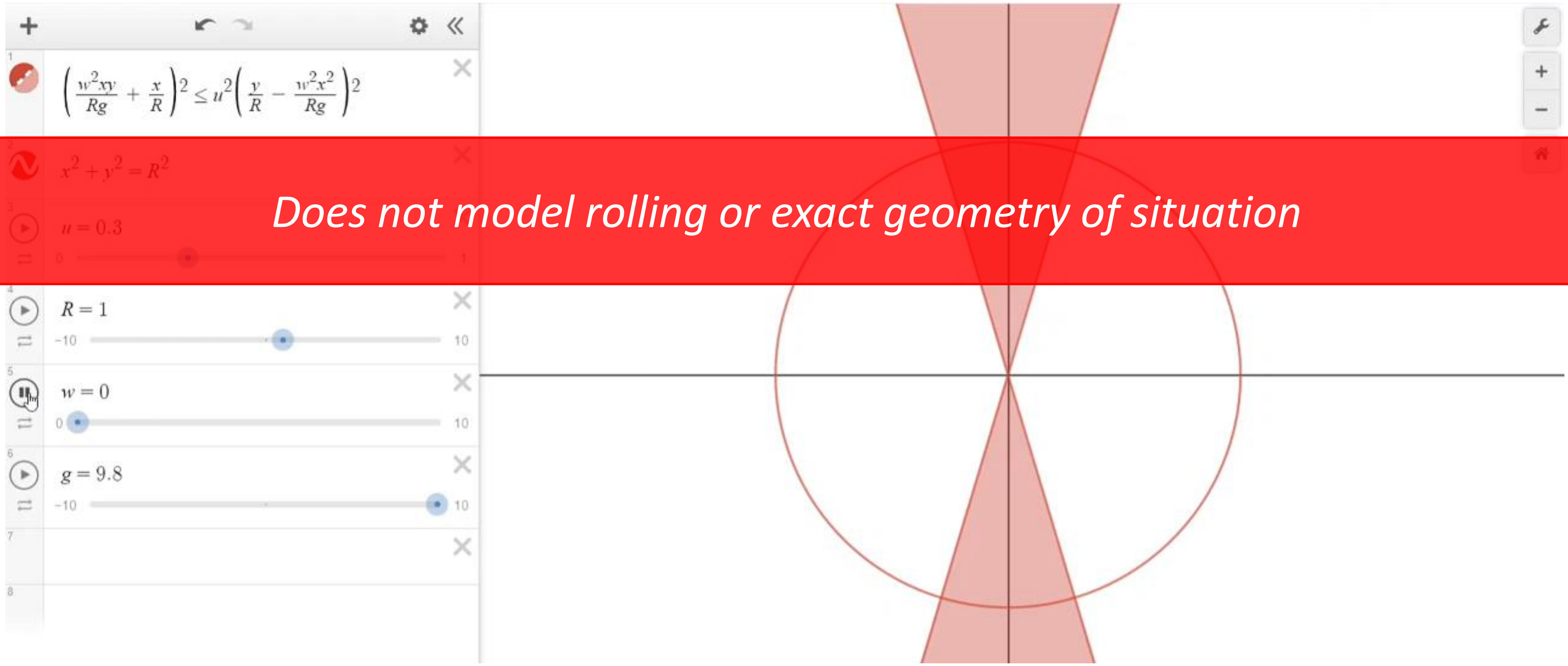
$$\left(\frac{\omega^2 yz}{Rg} + \frac{y}{R}\right)^2 \leq \mu^2\left(\frac{z}{R} - \frac{\omega^2 y^2}{Rg}\right)^2, \qquad y^2 + z^2 = R^2$$

(Balandin and Shalimova, 2015)

Stability Visualization

We can visualize this stability using Desmos in the animation below for the parameters:

$$\mu = 0.3, R = 1, 0 \le \omega \le 10, g = 9.8$$



1 $\left(\frac{w^2xy}{Rg}+\frac{x}{R}\right)^2 \le u^2\left(\frac{y}{R}-\frac{w^2x^2}{Rg}\right)^2$

2 $x^2+y^2=R^2$

3 $u=0.3$

4 $R=1$

5 $w=0$

6 $g=9.8$

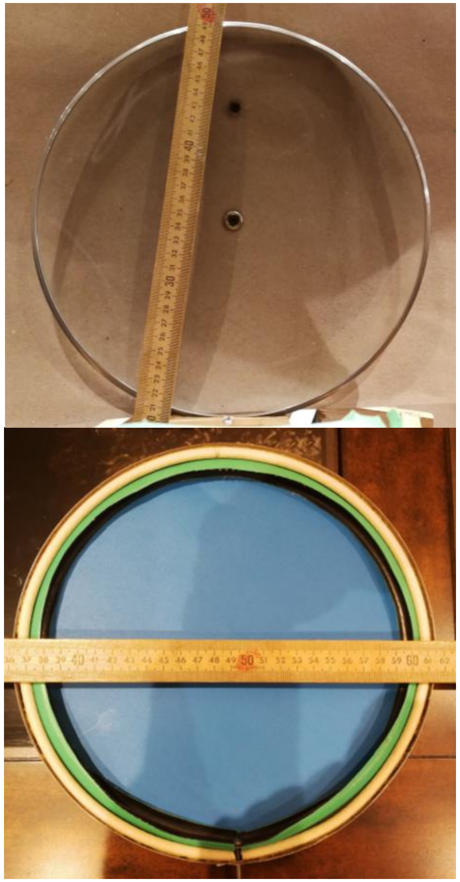
Does not model rolling or exact geometry of situation

Lagrangian Analysis

Axis of Rotation

Inclination and Resonance

Measuring Physical Parameters

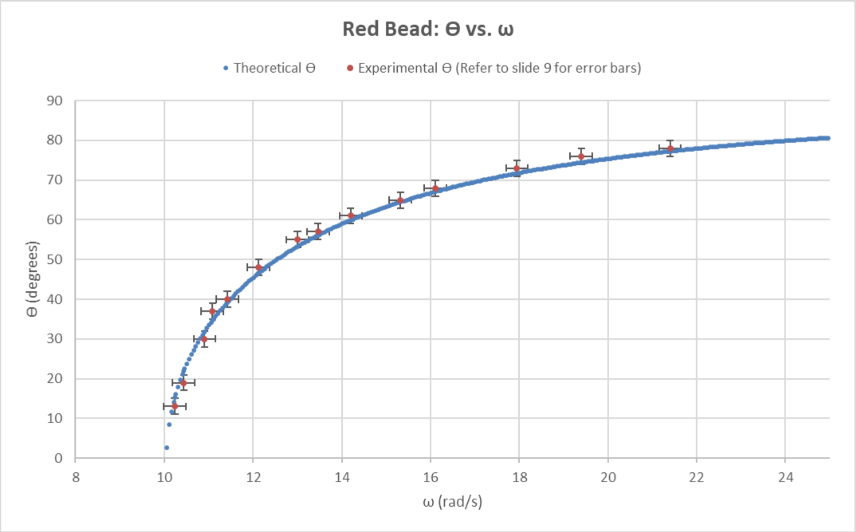


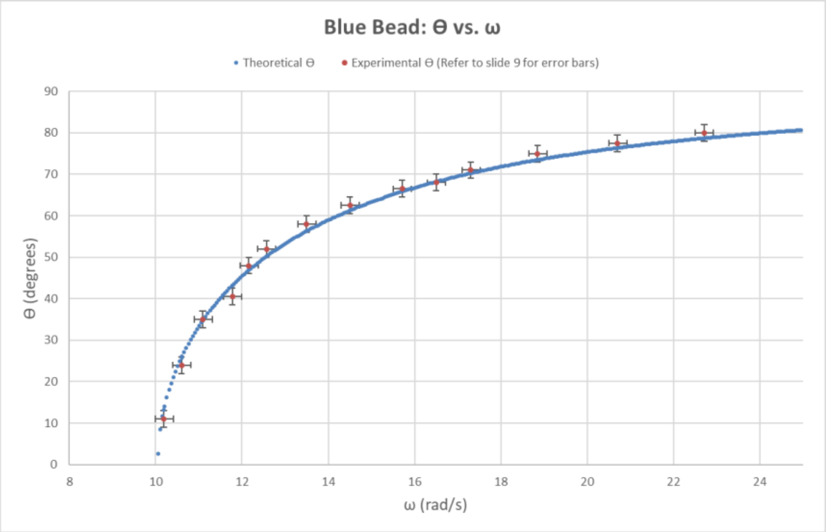
Meter Stick
±0.01 m

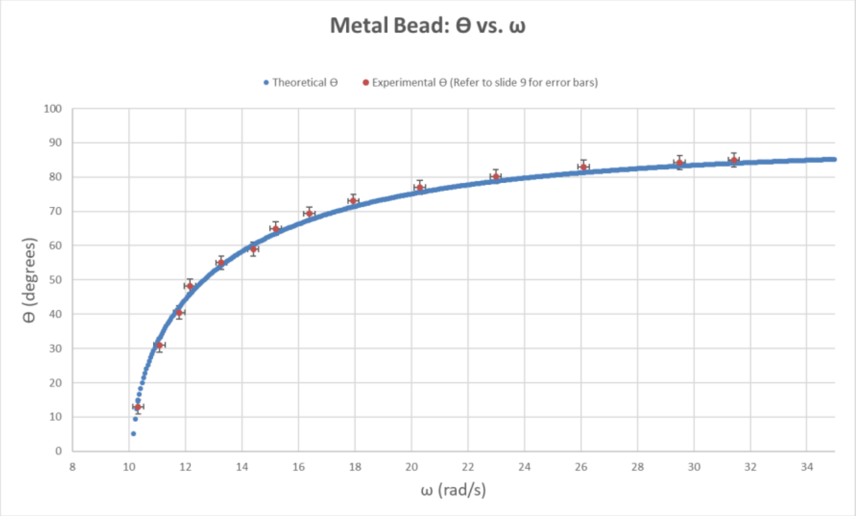
Vernier Caliper
±0.0002 m

Analytical Balance
±0.01 g

Varying ω and the Bead(Size and Mass) cont.





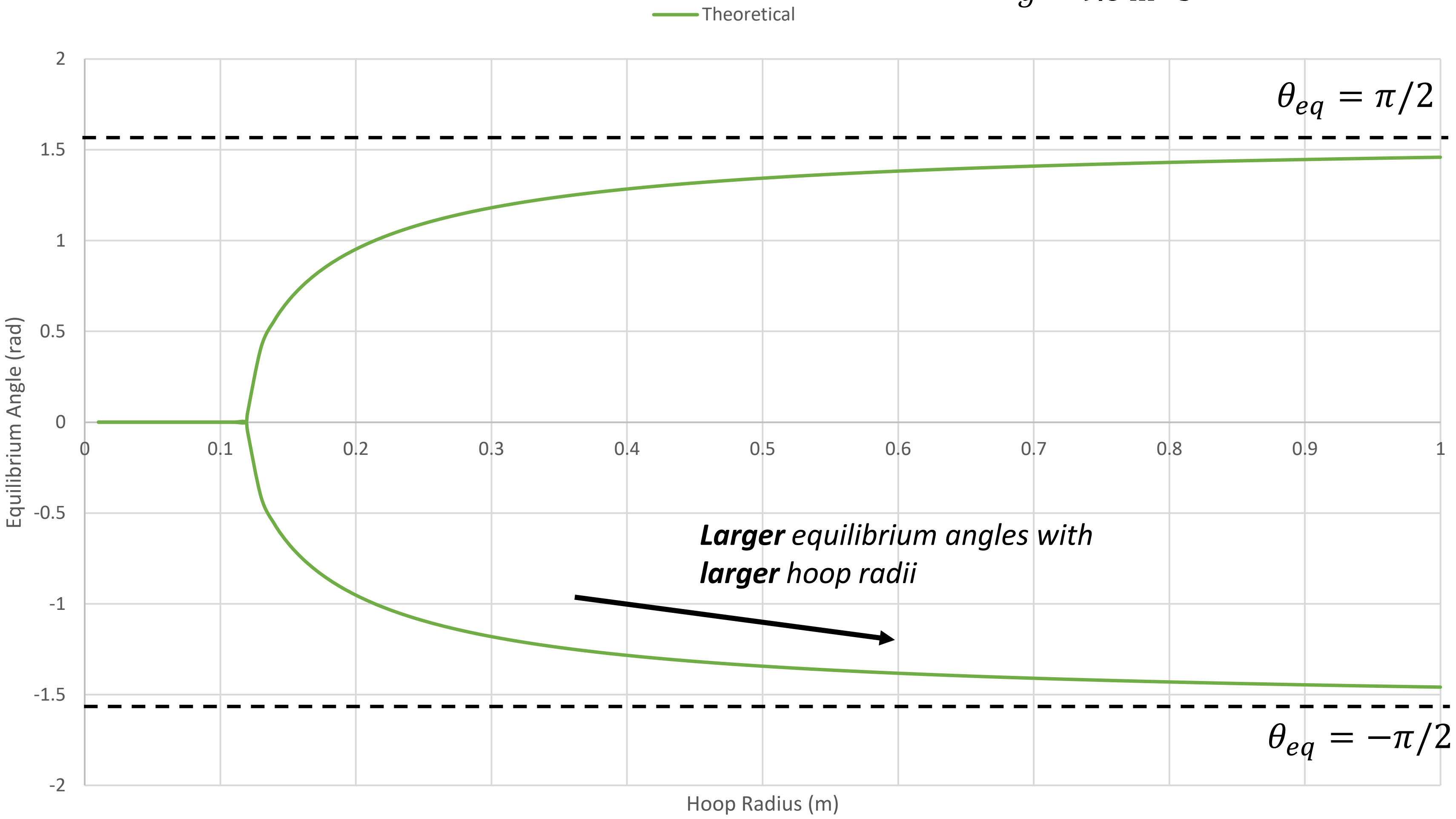


Blue & Red: *Since these beads behave similarly, therefore, the mass of the bead does not affect its motion.*

All: *Despite the beads having different textures, masses, and sizes, their motion is very similar. Those factors have an almost negligible effect on the outcome.*

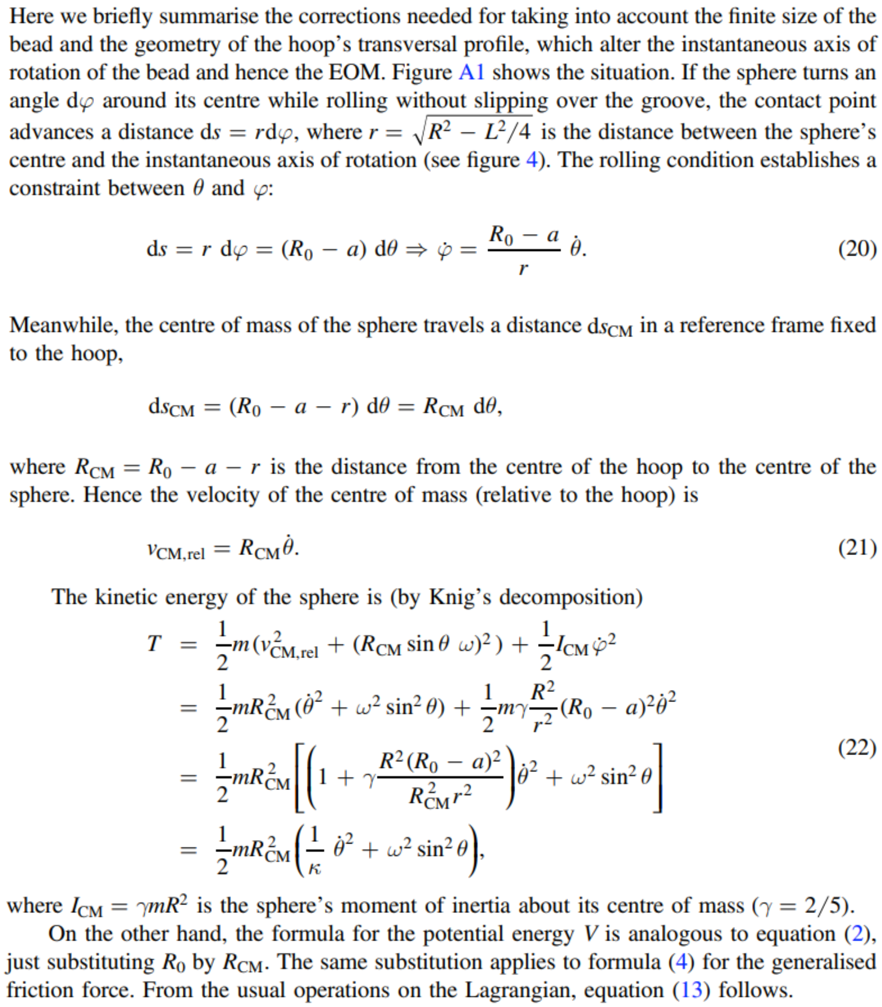
Varying Hoop Radius

$r = 0.0095\ \mathrm{m}$
$\omega = 9.42\ \mathrm{rad} \cdot \mathrm{s}^{-1}$
$g = 9.8\ \mathrm{m} \cdot \mathrm{s}^{-2}$



Larger *equilibrium angles with*
larger *hoop radii*

Appendix A (Raviola et al.)



Here we briefly summarise the corrections needed for taking into account the finite size of the bead and the geometry of the hoop's transversal profile, which alter the instantaneous axis of rotation of the bead and hence the EOM. Figure A1 shows the situation. If the sphere turns an angle $\mathrm{d}\varphi$ around its centre while rolling without slipping over the groove, the contact point advances a distance $\mathrm{d}s = r\mathrm{d}\varphi$, where $r = \sqrt{R^2 - L^2/4}$ is the distance between the sphere's centre and the instantaneous axis of rotation (see figure 4). The rolling condition establishes a constraint between θ and φ:

$$\mathrm{d}s = r\ \mathrm{d}\varphi = (R_0 - a)\ \mathrm{d}\theta \Rightarrow \dot{\varphi} = \frac{R_0 - a}{r}\ \dot{\theta}. \tag{20}$$

Meanwhile, the centre of mass of the sphere travels a distance $\mathrm{d}s_{\mathrm{CM}}$ in a reference frame fixed to the hoop,

$$\mathrm{d}s_{\mathrm{CM}} = (R_0 - a - r)\ \mathrm{d}\theta = R_{\mathrm{CM}}\ \mathrm{d}\theta,$$

where $R_{\mathrm{CM}} = R_0 - a - r$ is the distance from the centre of the hoop to the centre of the sphere. Hence the velocity of the centre of mass (relative to the hoop) is

$$v_{\mathrm{CM,rel}} = R_{\mathrm{CM}}\dot{\theta}. \tag{21}$$

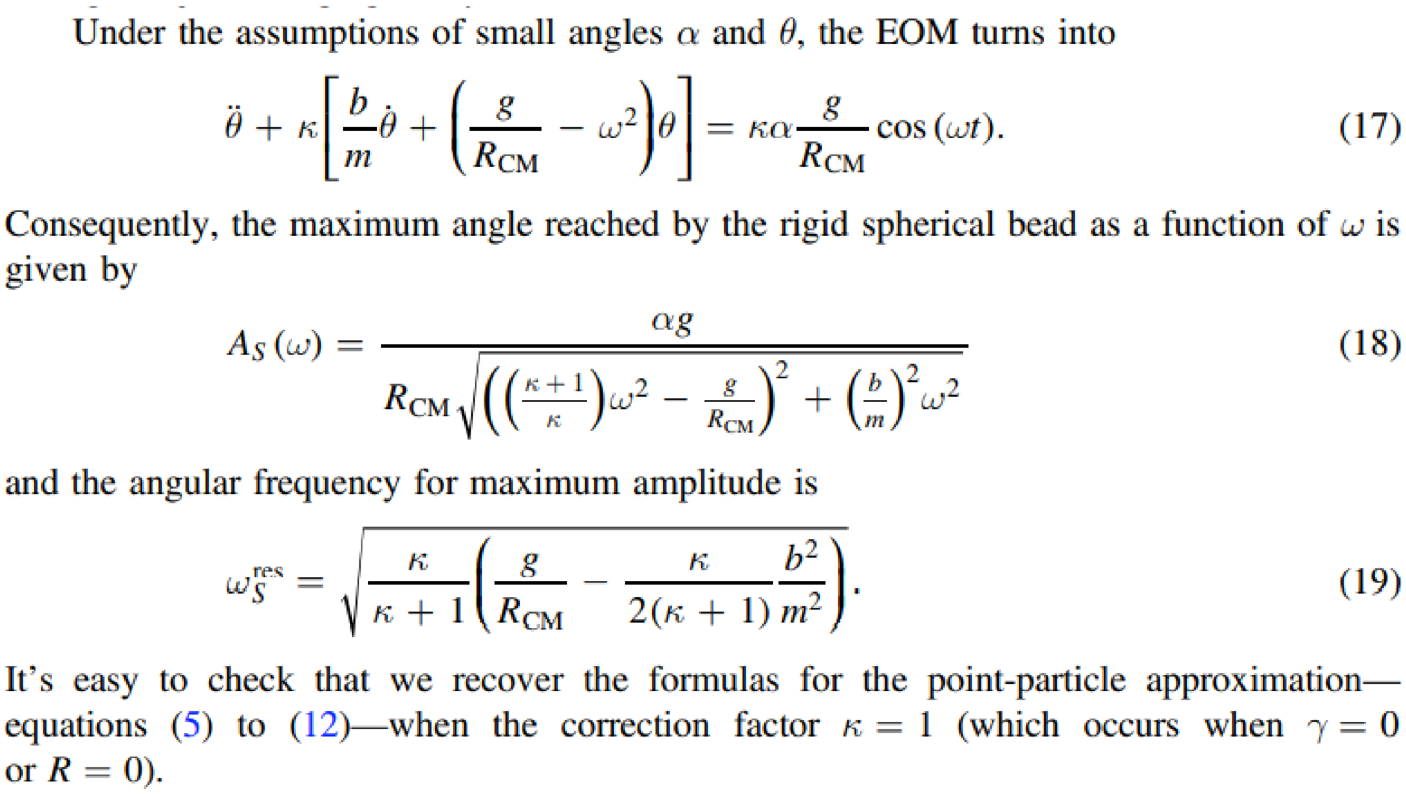
The kinetic energy of the sphere is (by Knig's decomposition)

$$\begin{aligned} T &= \frac{1}{2}m(v_{\mathrm{CM,rel}}^2 + (R_{\mathrm{CM}}\sin\theta\ \omega)^2) + \frac{1}{2}I_{\mathrm{CM}}\dot{\varphi}^2 \\ &= \frac{1}{2}mR_{\mathrm{CM}}^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + \frac{1}{2}m\gamma\frac{R^2}{r^2}(R_0 - a)^2\dot{\theta}^2 \\ &= \frac{1}{2}mR_{\mathrm{CM}}^2\left[\left(1 + \gamma\frac{R^2(R_0 - a)^2}{R_{\mathrm{CM}}^2 r^2}\right)\dot{\theta}^2 + \omega^2\sin^2\theta\right] \\ &= \frac{1}{2}mR_{\mathrm{CM}}^2\left(\frac{1}{\kappa}\ \dot{\theta}^2 + \omega^2\sin^2\theta\right), \end{aligned} \tag{22}$$

where $I_{\mathrm{CM}} = \gamma mR^2$ is the sphere's moment of inertia about its centre of mass ($\gamma = 2/5$).

On the other hand, the formula for the potential energy V is analogous to equation (2), just substituting R_0 by R_{CM}. The same substitution applies to formula (4) for the generalised friction force. From the usual operations on the Lagrangian, equation (13) follows.

Appendix A (Raviola et al.)



Under the assumptions of small angles α and θ, the EOM turns into

$$\ddot{\theta} + \kappa\left[\frac{b}{m}\dot{\theta} + \left(\frac{g}{R_{\mathrm{CM}}} - \omega^2\right)\theta\right] = \kappa\alpha\frac{g}{R_{\mathrm{CM}}}\cos(\omega t). \tag{17}$$

Consequently, the maximum angle reached by the rigid spherical bead as a function of ω is given by

$$A_S(\omega) = \frac{\alpha g}{R_{\mathrm{CM}}\sqrt{\left(\left(\frac{\kappa+1}{\kappa}\right)\omega^2 - \frac{g}{R_{\mathrm{CM}}}\right)^2 + \left(\frac{b}{m}\right)^2\omega^2}} \tag{18}$$

and the angular frequency for maximum amplitude is

$$\omega_S^{\mathrm{res}} = \sqrt{\frac{\kappa}{\kappa+1}\left(\frac{g}{R_{\mathrm{CM}}} - \frac{\kappa}{2(\kappa+1)}\frac{b^2}{m^2}\right)}. \tag{19}$$

It's easy to check that we recover the formulas for the point-particle approximation—equations (5) to (12)—when the correction factor $\kappa = 1$ (which occurs when $\gamma = 0$ or $R = 0$).

Appendix B (Balandin and Shalimova)

The motion of a heavy particle, that is, a bead P of mass m threaded onto a hoop in the form of a circle of radius ℓ with its centre at the point O, is considered. The hoop rotates with constant angular velocity ω about an inclined axis lying in its plane and passing through its centre. The angle of inclination of the axis from the vertical is assumed to be constant and equal to α. A dry friction force with a coefficient of friction μ acts between the bead and the hoop.

The motion of the bead can be described using Lagrange's equations of the first kind in a moving coordinate system (MCS) associated with the hoop. Suppose $Oxyz$ is a right-handed triplet with origin at the centre of the hoop, the z axis of which is directed along its axis of rotation, the y axis is located in the plane of the hoop and the x axis is perpendicular to this plane (see Fig. 1).

In the MCS, the bead position P is given by the coordinates (x, y, z) and the constraints restricting its motion are defined by the relations

$$f_1 = \frac{1}{2}(y^2 + z^2 - \ell^2) = 0, \quad f_2 = x = 0 \tag{1.1}$$

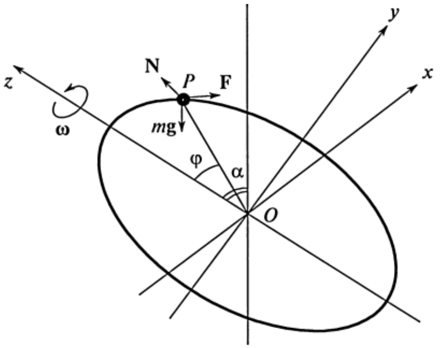


Fig. 1.

Suppose $\boldsymbol{v}_r = (\dot{x}, \dot{y}, \dot{z})$ is the bead velocity in the MCS, $\upsilon_r = (\upsilon_r, \upsilon_r)^{1/2}$, and the transfer velocity $\upsilon_e = (-\omega y, \omega x, 0)$. The kinetic energy of the system, free from constraints, and the potential energy in the MSC are given by the relations

$$T = \frac{1}{2}m((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2)$$

$$U = mg(x\sin\omega t\sin\alpha + y\cos\omega t\sin\alpha + z\cos\alpha)$$

Appendix B (Balandin and Shalimova)

where g is the gravitational acceleration. Lagrange's equations

$$\frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{x}} = \frac{\partial L_\lambda}{\partial x}, \quad \frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{y}} = \frac{\partial L_\lambda}{\partial y} + F_y, \quad \frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{z}} = \frac{\partial L_\lambda}{\partial z} + F_z \tag{1.2}$$

where

$$L_\lambda = L + \lambda_1 f_1 + \lambda_2 f_2, \quad L = T - U \tag{1.3}$$

and $\mathbf{F} = (0, F_y, F_z)$ is the friction force, can be represented in the form

$$m\mathbf{a} = \mathbf{F}_C + \mathbf{F}_c + \mathbf{F}_N + \mathbf{N} + \mathbf{F}$$

Here $\mathbf{a}$ is the acceleration of the bead in the MSC, $\mathbf{F}_C$ and $\mathbf{F}_c$ are the Coriolis force and the centrifugal force, $\mathbf{F}_N$ is the gravitational force and $\mathbf{N}$ is the normal reaction of the hoop. The unit vectors

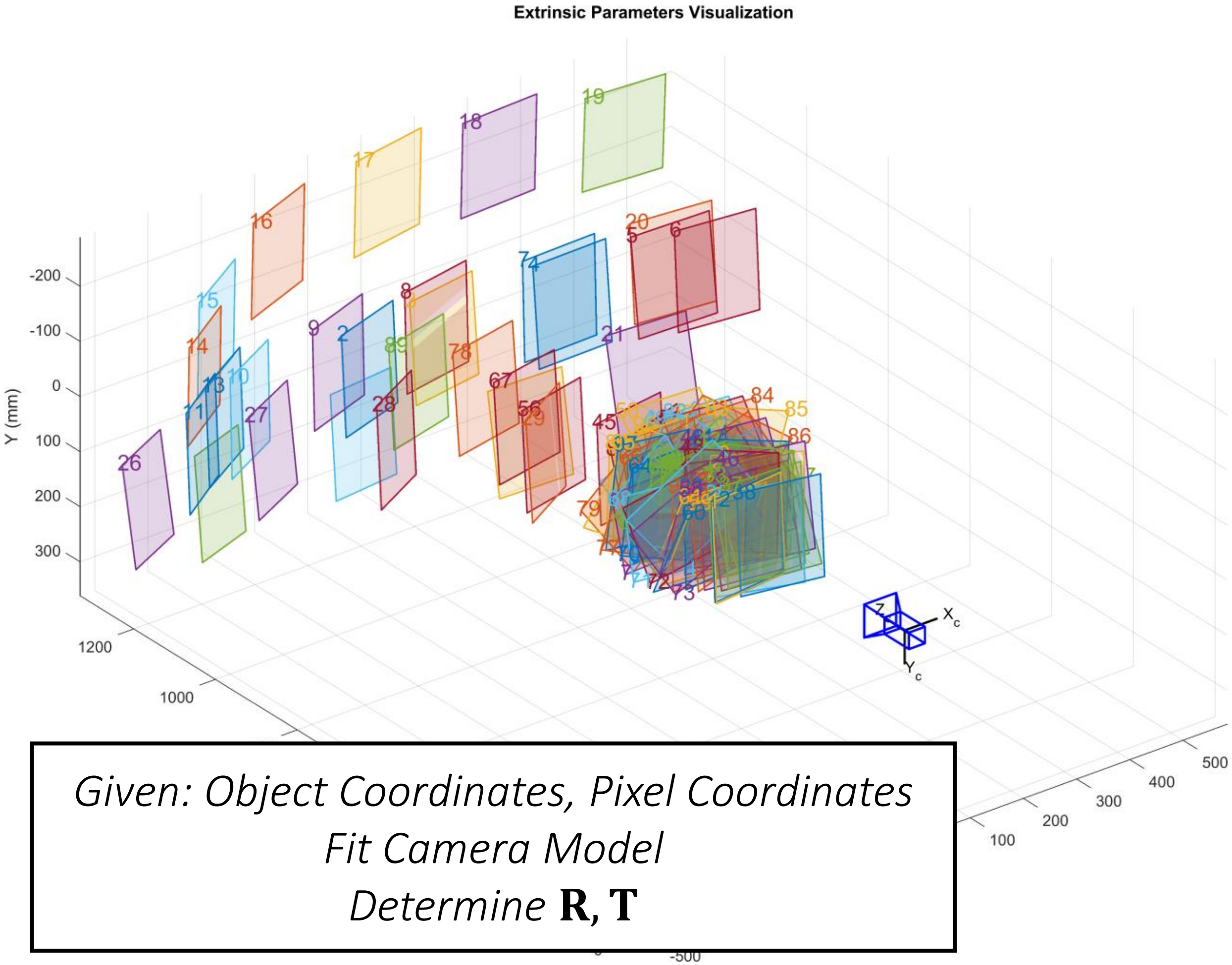
$$\boldsymbol{\tau} = \begin{pmatrix} 0 \\ -z/\ell \\ y/\ell \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 0 \\ -y/\ell \\ -z/\ell \end{pmatrix}, \quad \mathbf{b} = \boldsymbol{\tau} \times \mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

respectively define the tangent, internal normal and binormal to the circle at the point P. The expressions for the forces have the form

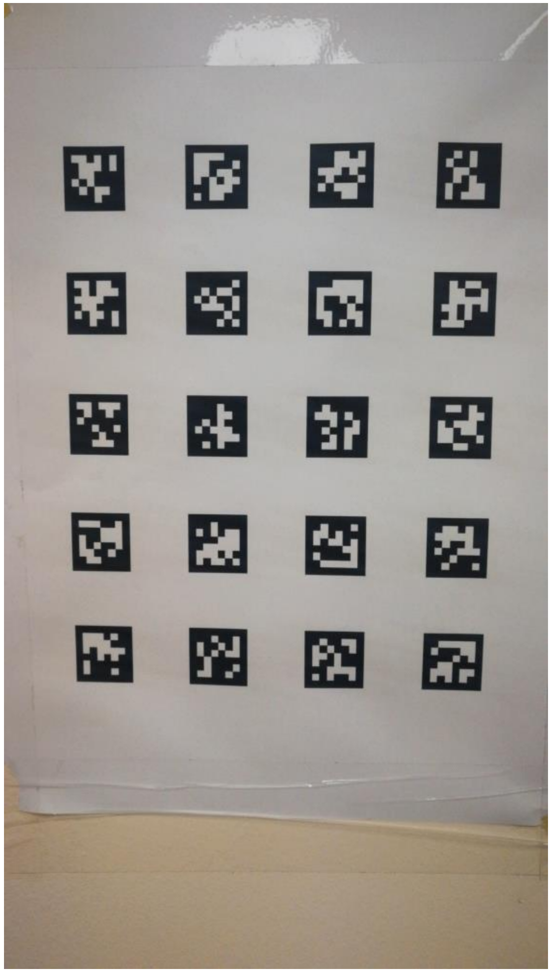
$$\mathbf{F}_C = \begin{pmatrix} 0 \\ -2m\omega\dot{y} \\ 2m\omega\dot{x} \end{pmatrix}, \quad \mathbf{F}_c = \begin{pmatrix} 0 \\ m\omega^2 y \\ m\omega^2 z \end{pmatrix}, \quad \mathbf{F}_N = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$\mathbf{N} = -\lambda_1 \ell \mathbf{n} + \lambda_2 \mathbf{b} = \begin{pmatrix} \lambda_2 \\ \lambda_1 y \\ \lambda_1 z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ F_y \\ F_z \end{pmatrix}, \quad \mathbf{F} \parallel \boldsymbol{\tau}$$

Camera Calibration



Given: Object Coordinates, Pixel Coordinates
Fit Camera Model
Determine $\mathbf{R}, \mathbf{T}$



Appendix B (Balandin and Shalimova)

and, in the case of slipping $\boldsymbol{v}_r \neq 0$

$$\mathbf{F} = -\mu \frac{\boldsymbol{v}_r}{v_r} N, \quad N = (\mathbf{N}, \mathbf{N})^{1/2} \tag{1.4}$$

We introduce dimensionless parameters using the relations

$$x \mapsto x\ell, \quad y \mapsto y\ell, \quad z \mapsto z\ell, \quad t \mapsto t\sqrt{\frac{\ell}{g}}, \quad \omega \mapsto \omega\sqrt{\frac{g}{\ell}}$$
$$\lambda_1 \mapsto \lambda_1 m\frac{g}{\ell}, \quad \lambda_2 \mapsto \lambda_2 mg, \quad L \mapsto Lmg\ell, \quad F_y \mapsto mgF_y, \quad F_z \mapsto mgF_z \tag{1.5}$$

Retaining a dot over a symbol as the notation for a derivative with respect to the new time and taking account of expression (1.5), the equations of constraints (1.1) and the Lagrangian (1.3) can be written as

$$f_1 = \frac{1}{2}(y^2 + z^2 - 1) = 0, \quad f_2 = x = 0$$
$$L_\lambda = \frac{1}{2}((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2)$$
$$- x\sin\omega t\sin\alpha - y\cos\omega t\sin\alpha - z\cos\alpha + \lambda_1 f_1 + \lambda_2 f_2 \tag{1.6}$$

To determine the Lagrange multipliers λ_1 and λ_2, it is necessary to calculate the first and second derivatives with respect to time of the identities specifying the constraints. These derivatives have the form

$$y\dot{y} + z\dot{z} = 0, \quad \dot{x} = 0,$$
$$y\ddot{y} + z\ddot{z} + \dot{y}^2 + \dot{z}^2 = 0, \quad \ddot{x} = 0 \tag{1.7}$$

Moreover, since the friction force vector touches the circle at the point P, we have

$$yF_y + zF_z = 0$$

and substitution of the expressions for the second derivatives from Eqs 1.2 into identities (1.7) enables us to represent λ_1 and λ_2 and also the equations of motion in the form

$$\lambda_1 = -(\dot{x}^2 + \dot{y}^2) - \omega^2 y^2 + y\cos\omega t\sin\alpha + z\cos\alpha$$
$$\lambda_2 = -2\omega\dot{y} + \sin\omega t\sin\alpha, \quad \ddot{y} = \omega^2 y - \cos\omega t\sin\alpha + \lambda_1 y + F_y$$
$$\ddot{z} = -\cos\alpha + \lambda_1 z + F_z \tag{1.8}$$

Appendix B (Balandin and Shalimova)

According to the Amontons–Coulomb law, the relation for the magnitude of the friction force

$$F^2 = F_y^2 + F_z^2 \le \mu^2(\lambda_1^2 + \lambda_2^2) \tag{1.9}$$

is satisfied.

Substitution of the expressions (1.8) for λ_1 and λ_2 into equality (1.9) gives the condition

$$F^2 \le \mu^2[(-(\dot{x}^2 + \dot{y}^2) - \omega^2 y^2 + y\cos\omega t\sin\alpha + z\cos\alpha)^2 + (-2\omega\dot{y} + \sin\omega t\sin\alpha)^2]$$

which becomes an equality in the case of sliding and, according to relation (1.4),

$$F_y = -F\frac{\dot{y}}{\sqrt{\dot{y}^2 + \dot{z}^2}}, \quad F_z = -F\frac{\dot{z}}{\sqrt{\dot{y}^2 + \dot{z}^2}}$$

$$F = \mu[(y\cos\omega t\sin\alpha + z\cos\alpha - (\dot{x}^2 + \dot{y}^2) - \omega^2 y^2)^2 + (\sin\omega t\sin\alpha - 2\omega\dot{y})^2]^{1/2}$$

since the direction of the friction force is opposite to the direction of sliding.

In the case of equilibrium of the bead with respect to the hoop, the inequality

$$F^2 \le \mu^2(y\cos\omega t\sin\alpha + z\cos\alpha - \omega^2 y^2)^2 + \mu^2\sin^2\omega t\sin^2\alpha \tag{1.10}$$

must be satisfied for all instants *t*.

Appendix B (Balandin and Shalimova)

If the bead is in equilibrium with respect to the hoop, its relative velocity and, together with it, also the Coriolis force, are equal to zero. The friction force, acting along the tangent to the hoop then compensates the sum of the tangential components of the gravitational force and the centrifugal force, that is,

$$-z(\omega^2 y - \cos\omega t\sin\alpha - Fz) + y(-\cos\alpha + Fy) = 0 \tag{2.1}$$

We now introduce the notation

$$\xi(y, z) = \omega^2 yz - z\cos\omega t\sin\alpha + y\cos\alpha$$

$$\eta(y, z) = (y\cos\omega t\sin\alpha + z\cos\alpha - \omega^2 y^2)^2 + \sin^2\omega t\sin^2\alpha$$

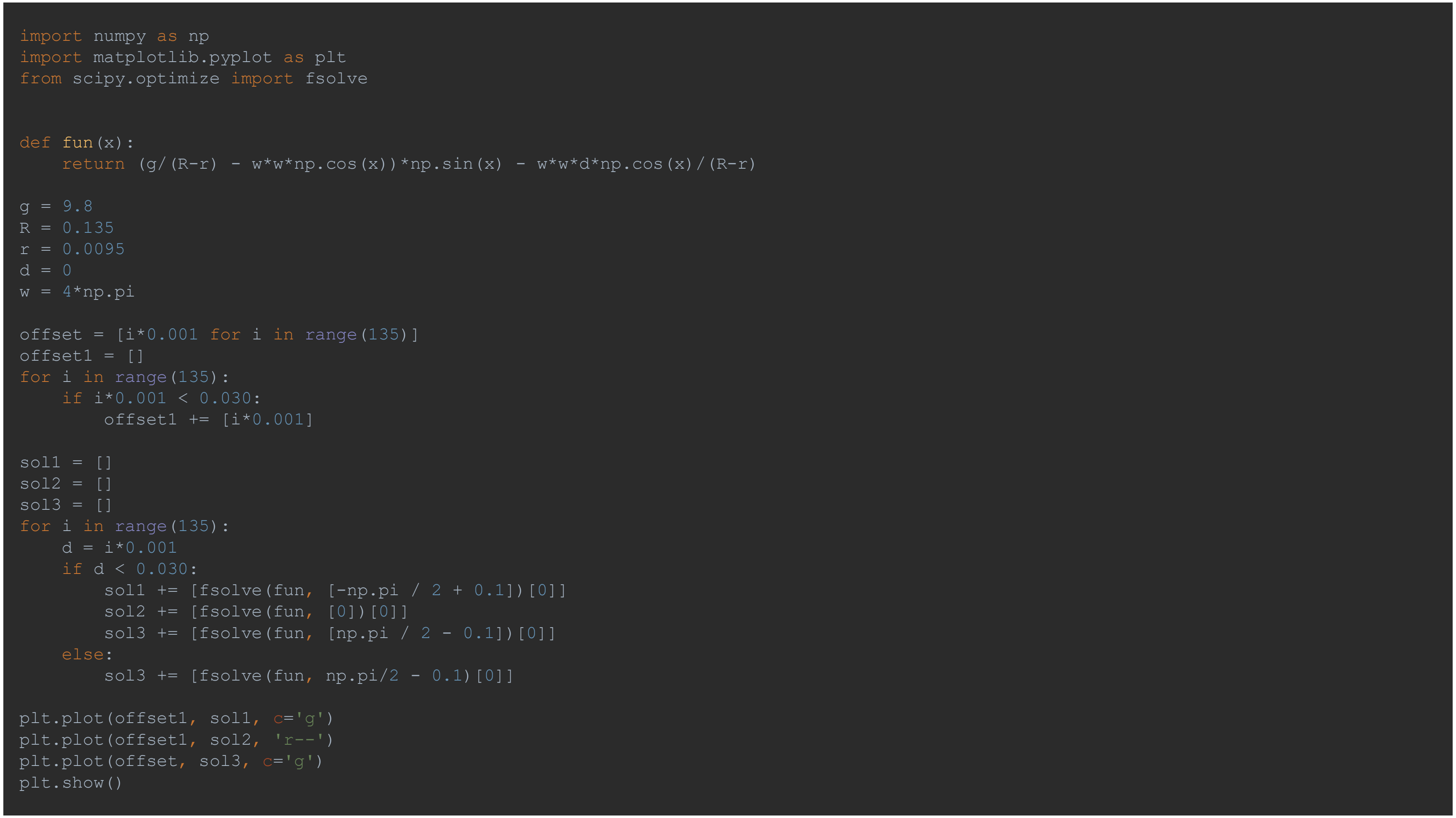
Using the first equality of (1.6), Eq. (2.1) is reduced to the form

$$F = \omega^2 yz - z\cos\omega t\sin\alpha + y\cos\alpha \tag{2.2}$$

This equation together with the first equality of (1.6) and inequality (1.10) forms a system for determining the relative equilibria and, after substituting expression (2.2) for F, this system takes the form

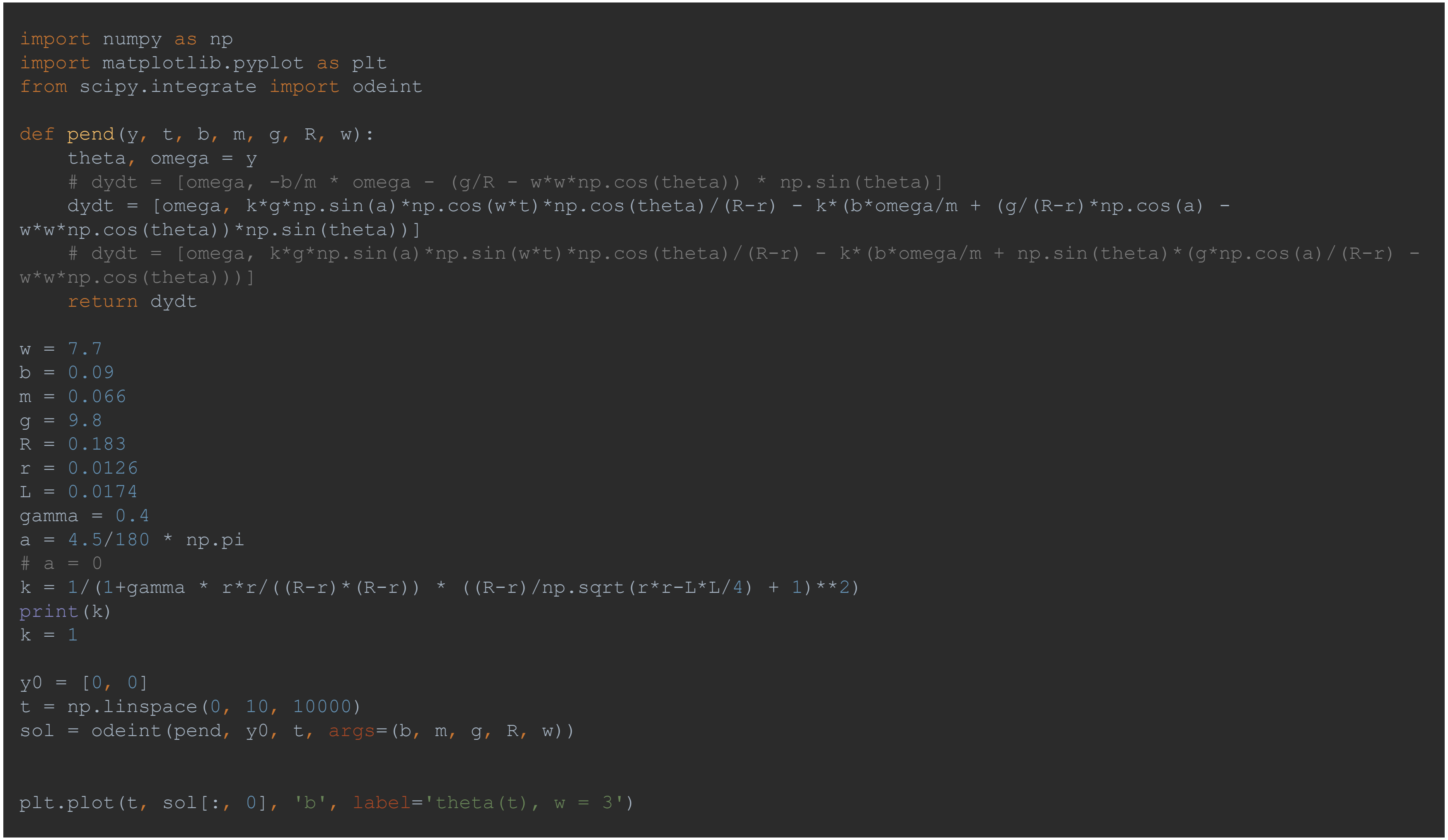
$$\xi^2(y, z) \le \mu^2\eta(y, z), \quad y^2 + z^2 = 1 \tag{2.3}$$

Appendix C (Axis Offset Code)



```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve


def fun(x):
    return (g/(R-r) - w*w*np.cos(x))*np.sin(x) - w*w*d*np.cos(x)/(R-r)

g = 9.8
R = 0.135
r = 0.0095
d = 0
w = 4*np.pi

offset = [i*0.001 for i in range(135)]
offset1 = []
for i in range(135):
    if i*0.001 < 0.030:
        offset1 += [i*0.001]

sol1 = []
sol2 = []
sol3 = []
for i in range(135):
    d = i*0.001
    if d < 0.030:
        sol1 += [fsolve(fun, [-np.pi / 2 + 0.1])[0]]
        sol2 += [fsolve(fun, [0])[0]]
        sol3 += [fsolve(fun, [np.pi / 2 - 0.1])[0]]
    else:
        sol3 += [fsolve(fun, np.pi/2 - 0.1)[0]]

plt.plot(offset1, sol1, c='g')
plt.plot(offset1, sol2, 'r--')
plt.plot(offset, sol3, c='g')
plt.show()
```

Appendix D (ODE Solver Code)



```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

def pend(y, t, b, m, g, R, w):
    theta, omega = y
    # dydt = [omega, -b/m * omega - (g/R - w*w*np.cos(theta)) * np.sin(theta)]
    dydt = [omega, k*g*np.sin(a)*np.cos(w*t)*np.cos(theta)/(R-r) - k*(b*omega/m + (g/(R-r)*np.cos(a) -
w*w*np.cos(theta))*np.sin(theta))]
    # dydt = [omega, k*g*np.sin(a)*np.sin(w*t)*np.cos(theta)/(R-r) - k*(b*omega/m + np.sin(theta)*(g*np.cos(a)/(R-r) -
w*w*np.cos(theta)))]
    return dydt

w = 7.7
b = 0.09
m = 0.066
g = 9.8
R = 0.183
r = 0.0126
L = 0.0174
gamma = 0.4
a = 4.5/180 * np.pi
# a = 0
k = 1/(1+gamma * r*r/((R-r)*(R-r)) * ((R-r)/np.sqrt(r*r-L*L/4) + 1)**2)
print(k)
k = 1

y0 = [0, 0]
t = np.linspace(0, 10, 10000)
sol = odeint(pend, y0, t, args=(b, m, g, R, w))


plt.plot(t, sol[:, 0], 'b', label='theta(t), w = 3')
```

Appendix E (Air Resistance)

Maximal Tangential Velocity: $\omega_{max} \cdot R \approx 3\ \mathrm{m/s}$

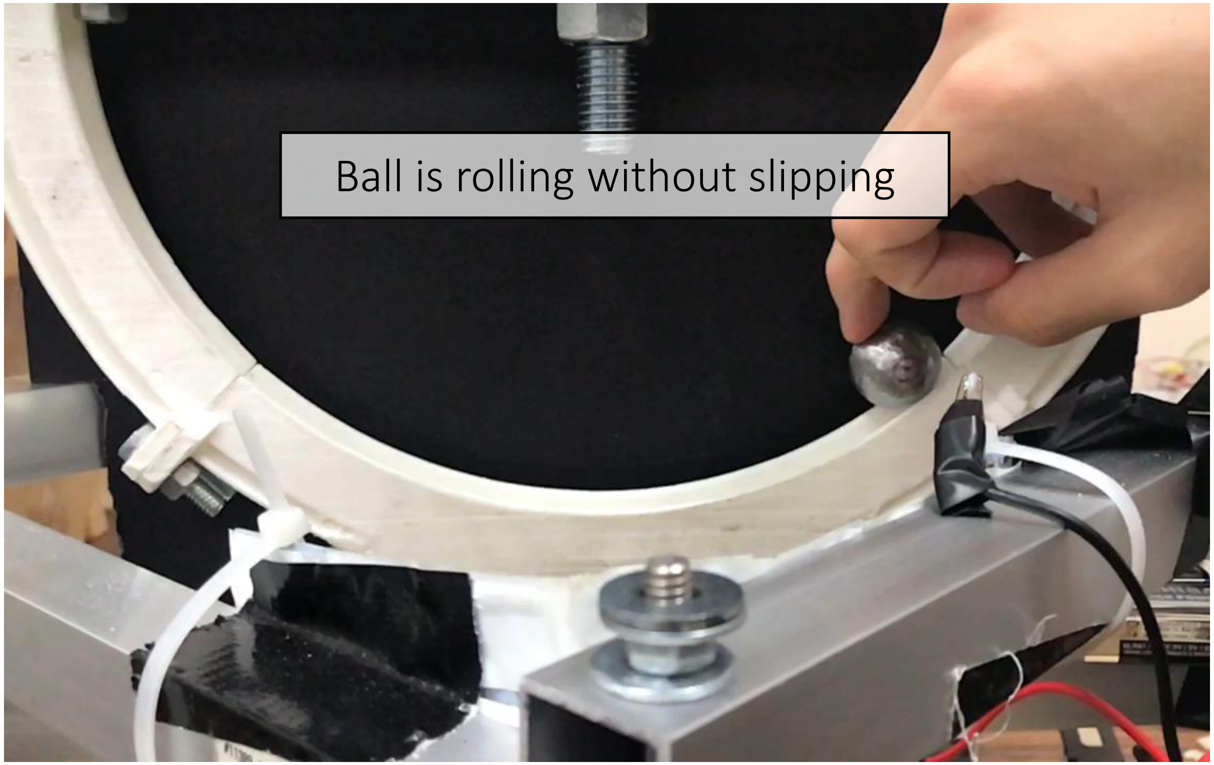
Maximal Tangential Velocity: $\dot{\theta}_{max} \cdot R \approx 0.3\ \mathrm{m/s}$

$$a_T = \frac{\rho v_T^2 A C_D}{2m} \approx 0.1\ \mathrm{m \cdot s^{-2}}$$

$$\boxed{a_R = \frac{\rho v_R^2 A C_D}{2m} \approx 0.001\ \mathrm{m \cdot s^{-2}}}$$

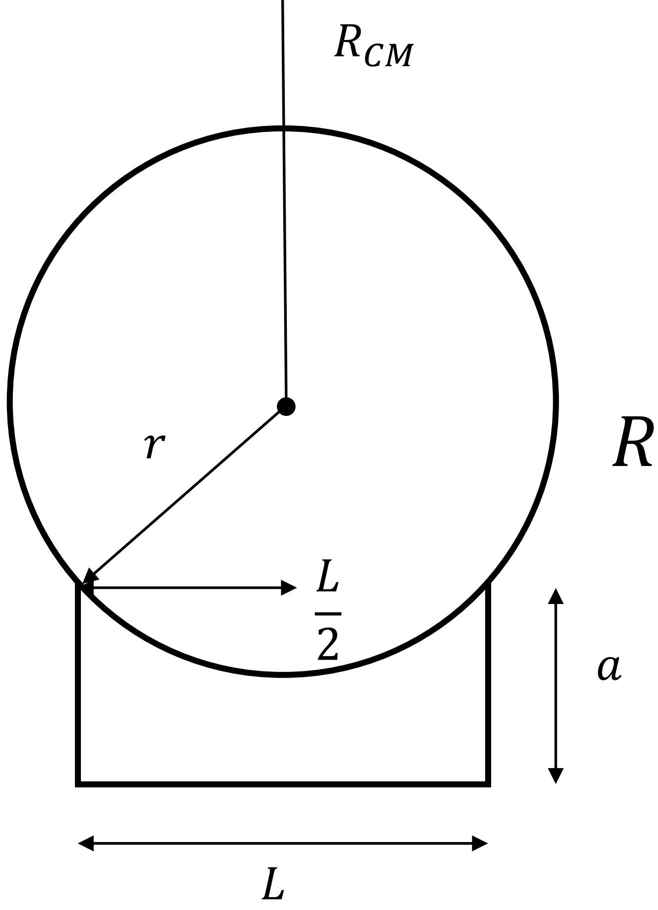
Radial acceleration due to air resistance is insignificant

Appendix F (No-slip)



Ball is rolling without slipping

Appendix G ($\mathbf{R_{CM}}$ Calculation)



$$R_{CM} = R - \left(a + \sqrt{r^2 - \left(\frac{L}{2}\right)^2}\right)$$

0.15-(0.015+sin(45)*0.0127)

Appendix H (Error on 3D Printer)

*The print accuracy of Anycubic Chiron is **0.05 to 0.3 mm***

*The positional accuracy of the printer in the **X** and **Y** axis is **0.0125 mm** while in the **Z** axis it is **0.0020 mm**.*

Appendix I (PID)

kP = proportional gain

kI = integral gain

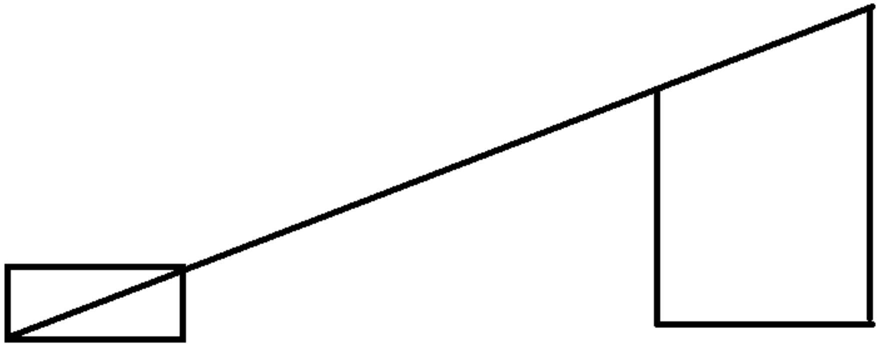
kD = derivative gain

Error (Angular Velocity Error) = DesiredAngularVelocity – currentAngularVelocity

$\text{Correction term} = (kP * \text{Error}) - (kD*(\text{Error}_n - \text{Error}_{n-1}) / \text{TimeInterval})$

Duty Cycle (pwm) = Duty Cycle + Correction Term

Appendix H (Calculation of Incline)



Appendix J (LSODA Algorithm)

High Accuracy low failure rate compared to other ODE solving methods

Appendix K

This is really a 3D problem. You have two normal forces that point towards the center of the bead and each of those normal forces has a component that is out of plane of the hoop. Even more, those two forces are rarely equal because if the bead is moving up, then one normal force is responsible for speeding it up tangentially and if the bead's angle is decreasing, then the other edges normal force has to slow it down.

Appendix L (Dynamics)

Higher Angular Velocity → Higher Equilibrium Angle → Higher frequency of Oscillation about Equilibrium

Decay of Bead

Constants:

Gravity

Radius

Mass

Moment of Inertia

Initial Conditions:

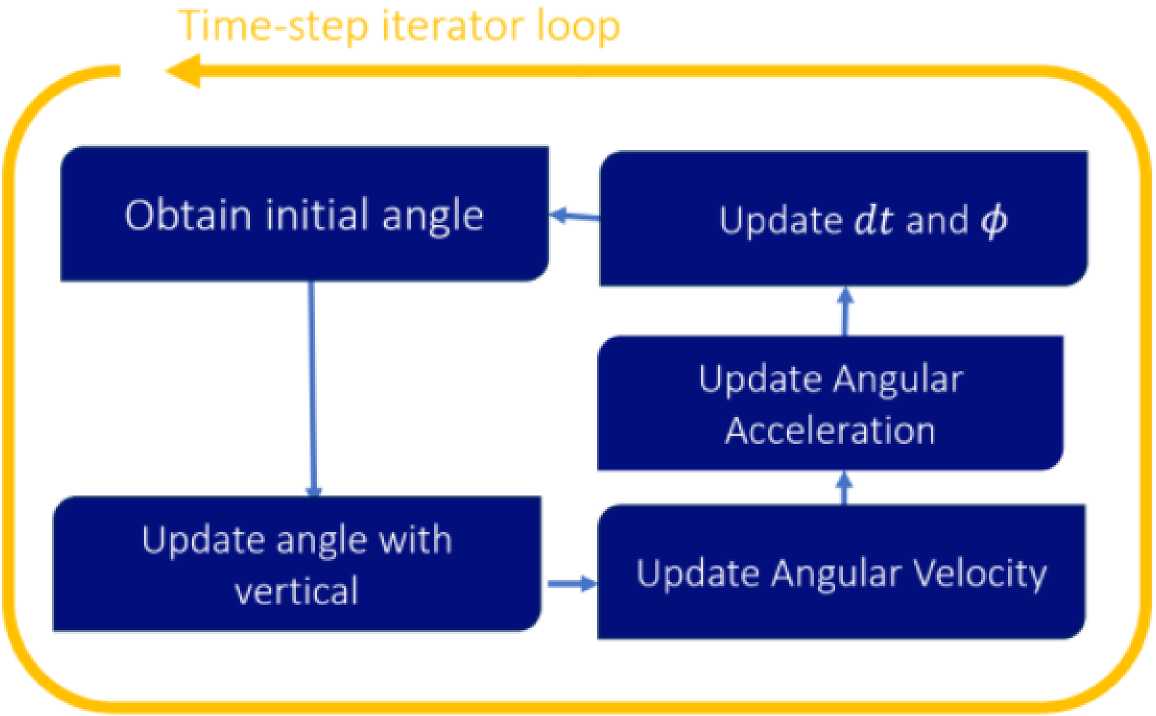
Time = 0

Angle with the Vertical = 0

Termination Condition:

Reaches Equilibrium/Converges

Decay stops



Time-step iterator loop

Obtain initial angle

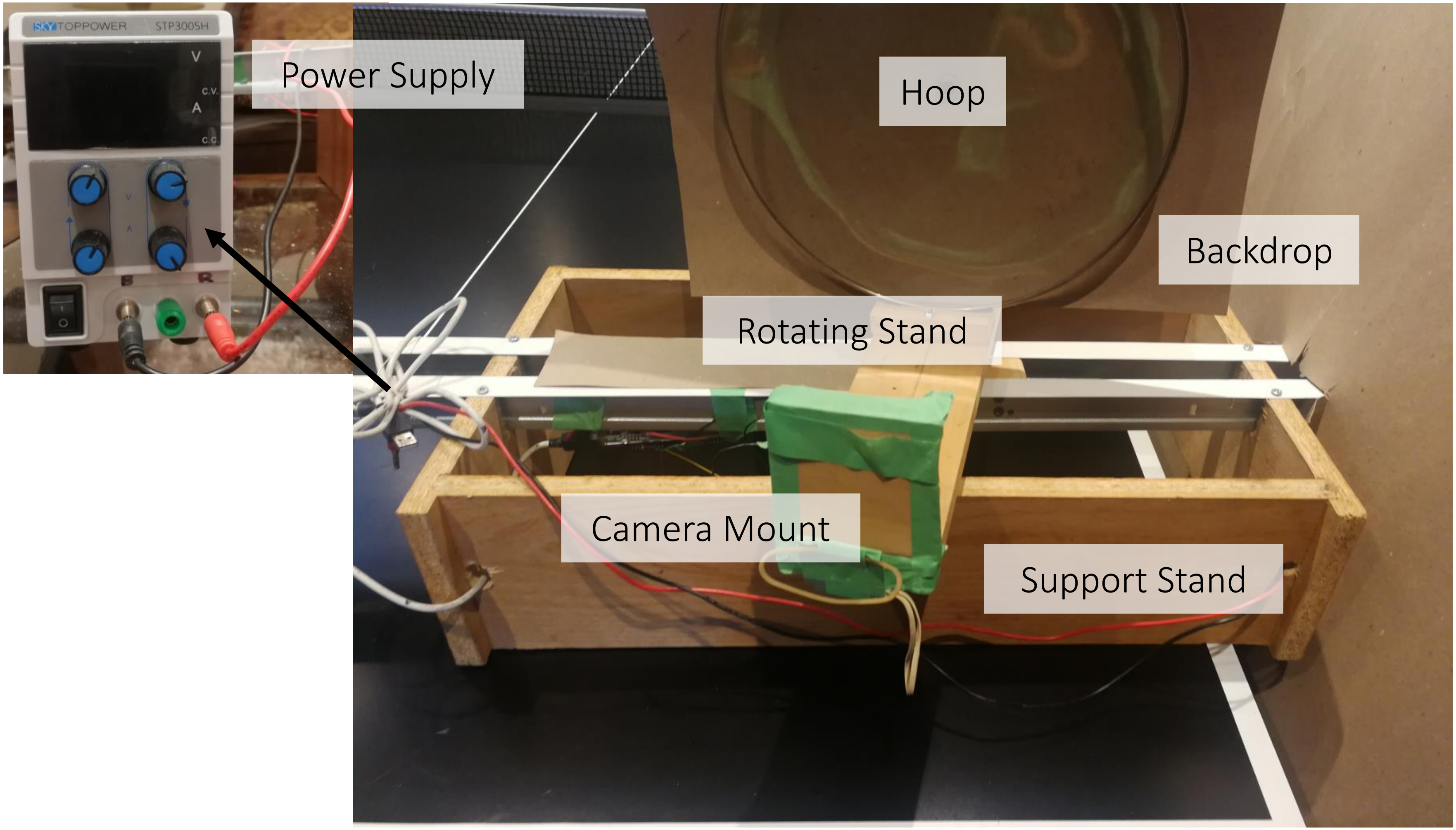
Update angle with vertical

Update Angular Velocity

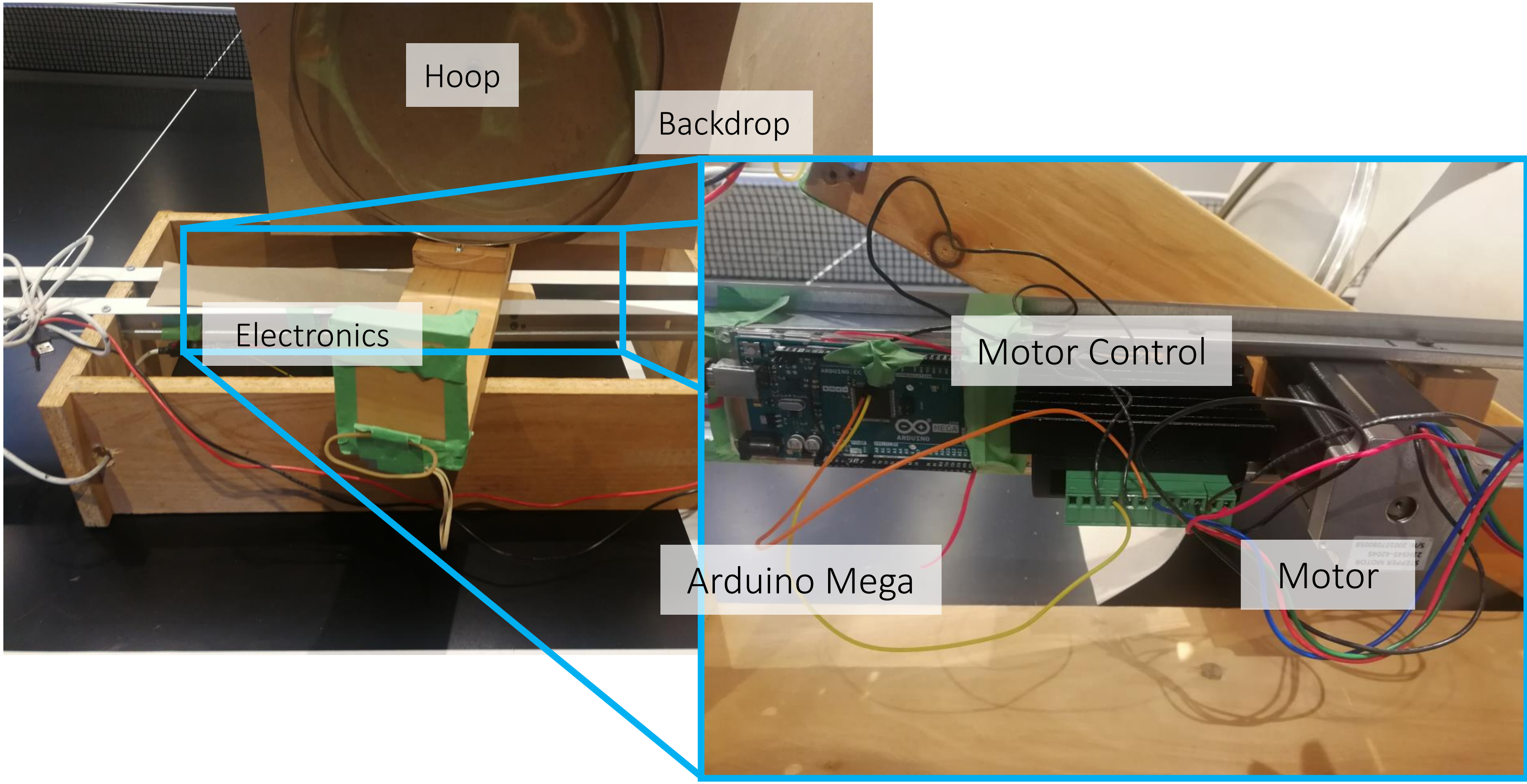
Update Angular Acceleration

Update dt and ϕ

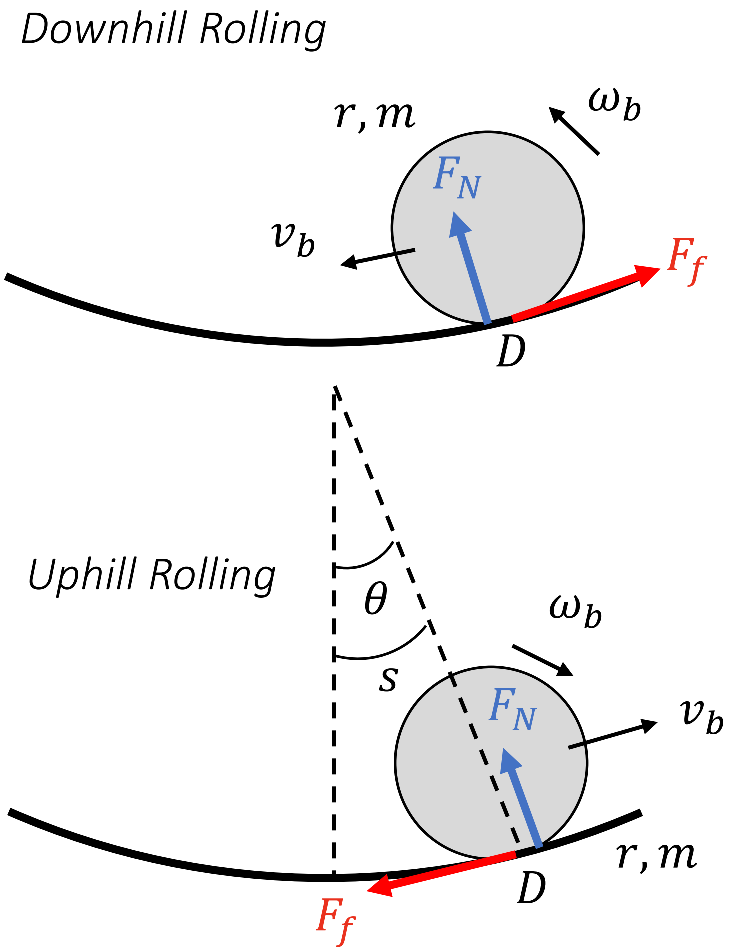
Experimental Setup



Experimental Setup



Rolling Friction



Equations of Motion

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -mg\sin\theta + F_f$$

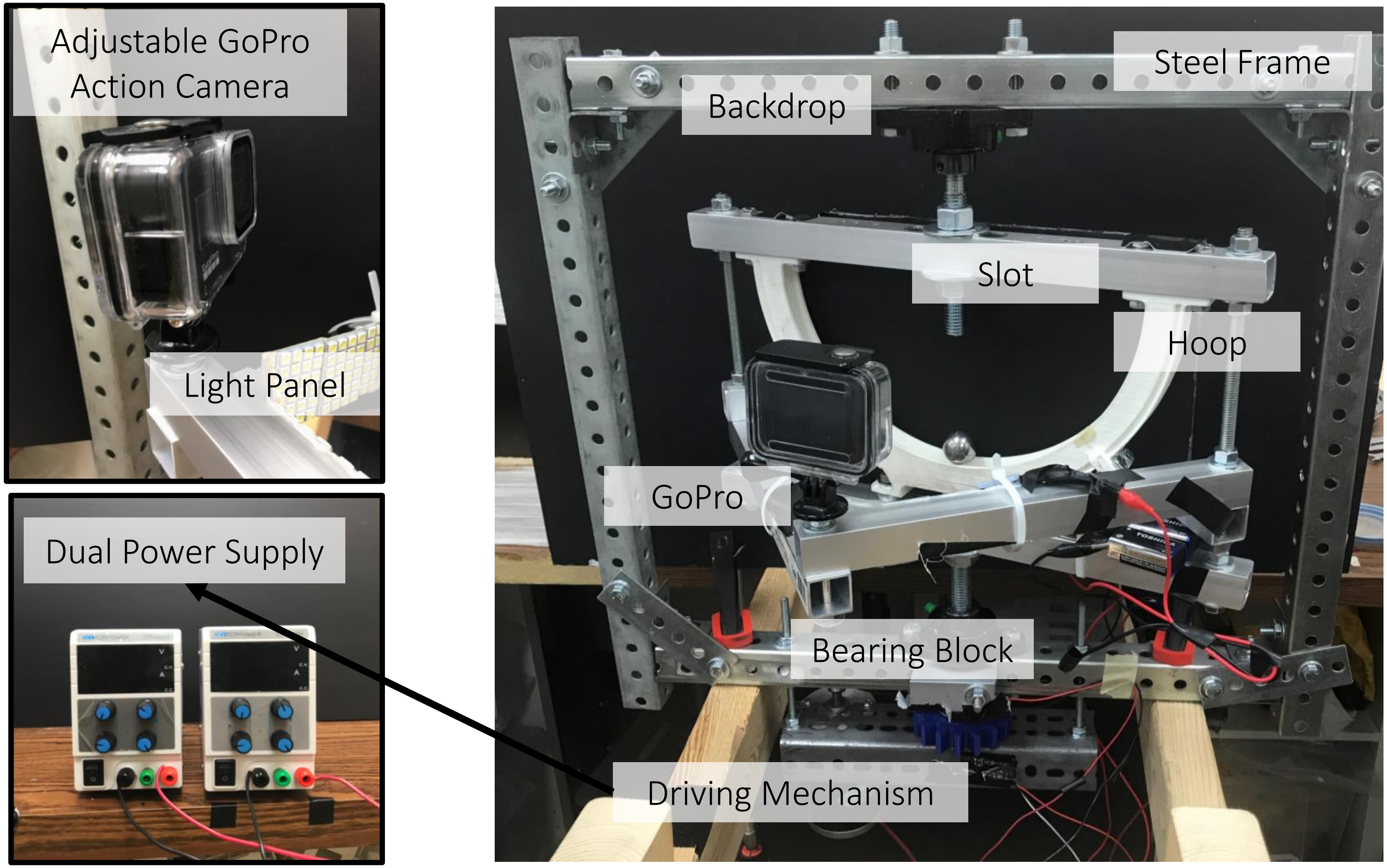
$$F_f r - F_N D = I_{cm}\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

$$\Big\downarrow$$

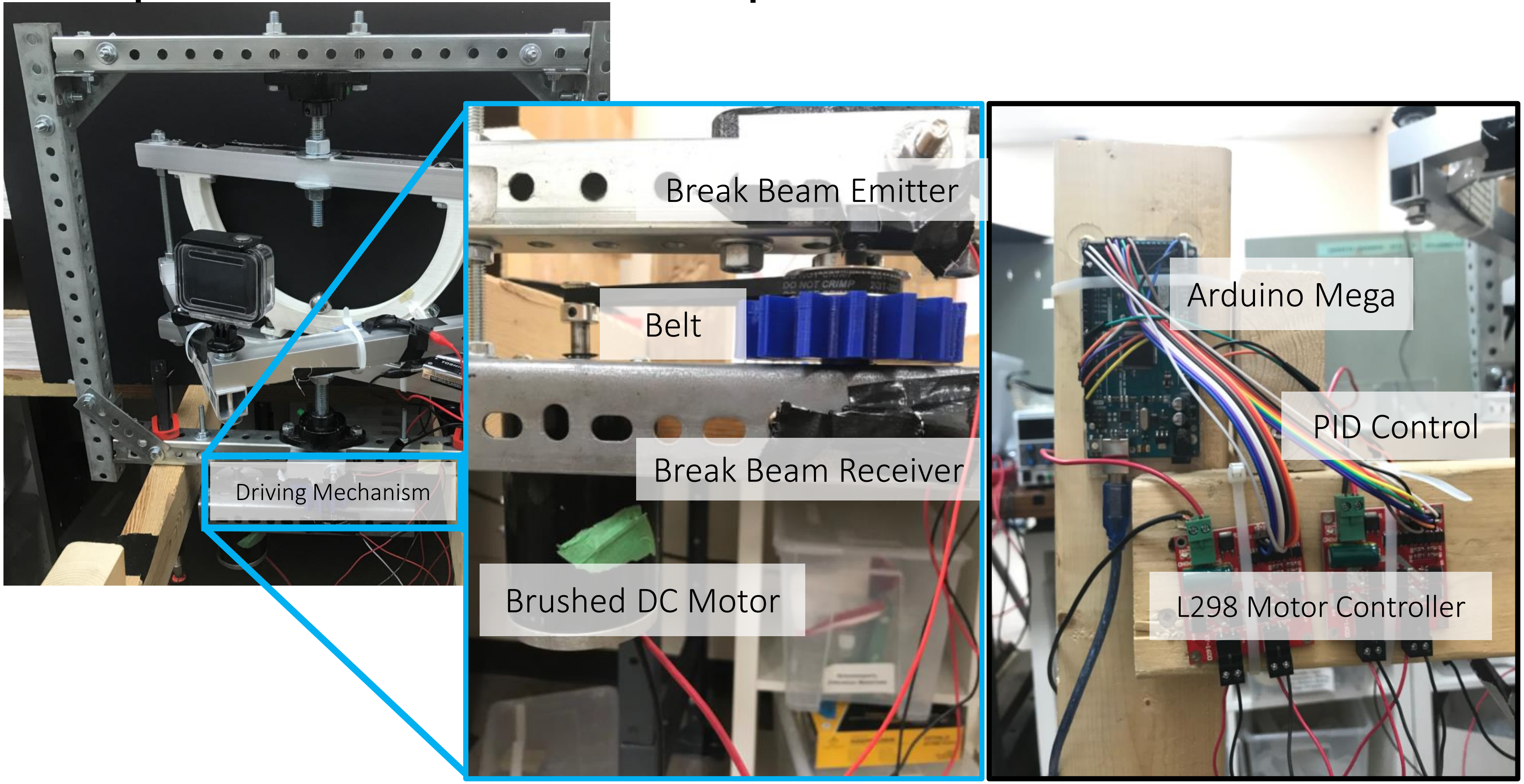
(Cross, 2016)

$$\mu_R = bv$$

Experimental Setup



Experimental Setup



Maximum angle reached by the rigid spherical bead as a function of different ω

$$A_s(\omega) = \frac{\alpha g}{R_{CM}\sqrt{\left(\left(\frac{k+1}{k}\right)\omega^2 - \frac{g}{R_{CM}}\right) + \left(\frac{b}{m}\right)^2 \omega^2}}$$

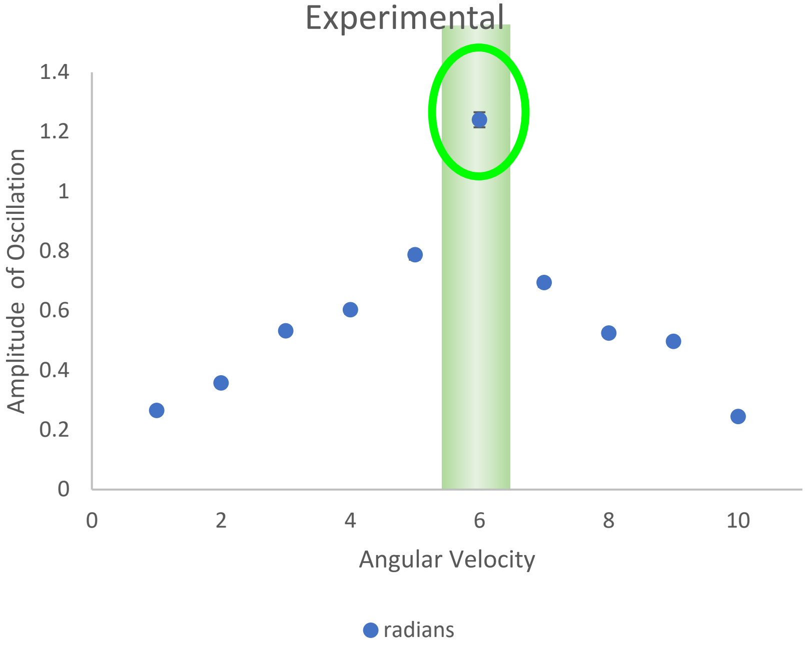
✓

Resonance

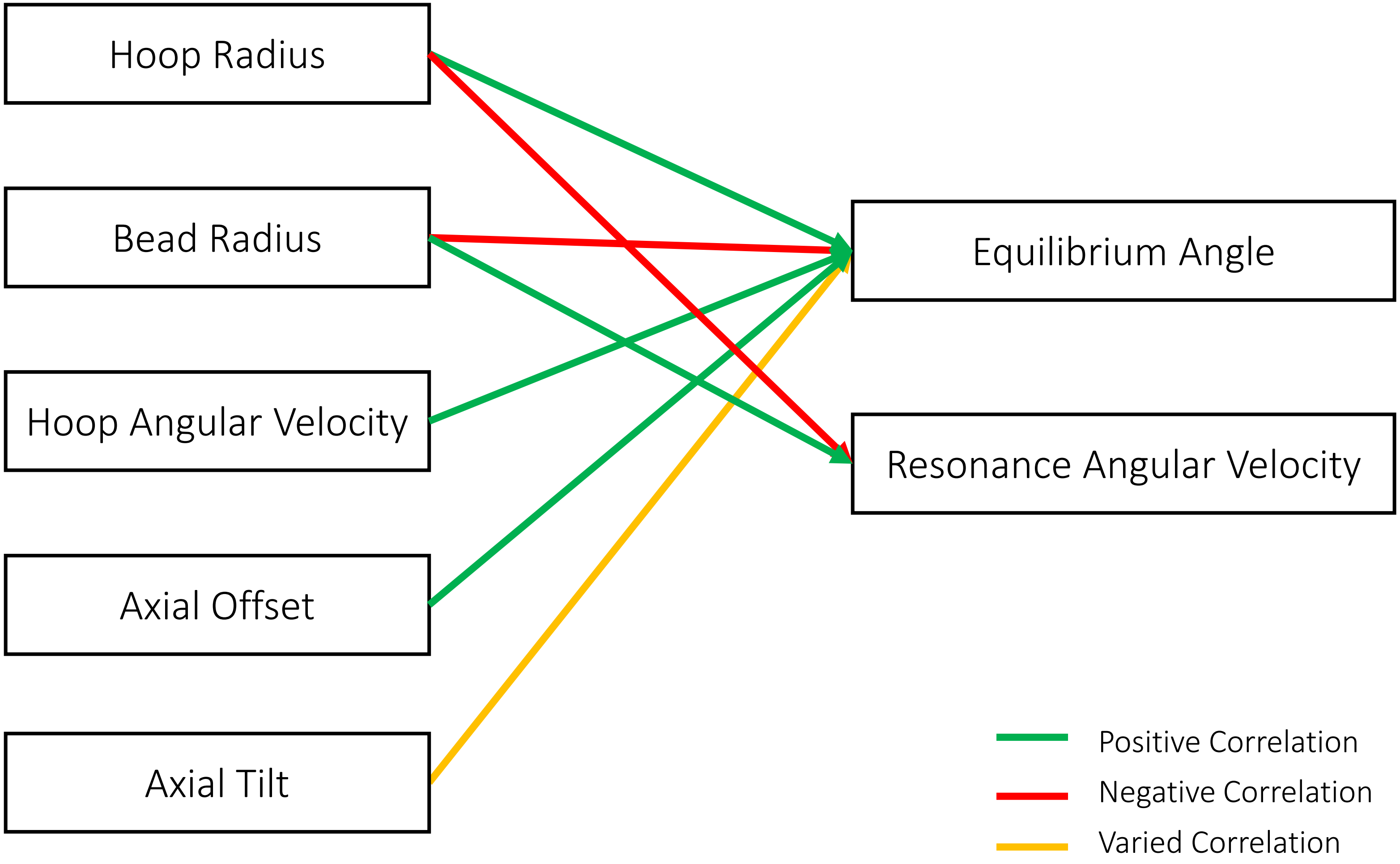
Resonance occurs at around 6 radians per second

Angular frequency for maximum amplitude is

$$\omega_S^{res} = \sqrt{\frac{k+1}{k}\left(\frac{g}{R_{CM}} - \frac{k}{2(k+1)}\frac{b^2}{m^2}\right)}$$



Key Parameters



Hoop Radius

Bead Radius

Hoop Angular Velocity

Axial Offset

Axial Tilt

Equilibrium Angle

Resonance Angular Velocity

🟩 Positive Correlation

🟥 Negative Correlation

🟨 Varied Correlation